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Topology and its Applications 78 (1997) 153–186

TOPOLOGY AND ITS APPLICATIONS

On the quotient of the braid group by commutators of transversal half-twists and its group actions ☆

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Received 25 December 1995; revised 21 May 1995, 25 July 1996

Abstract

This paper presents and describes a quotient of the Artin braid group by commutators of transversal half-twists and investigates its group actions. We denote the quotient by $\widetilde{B}_n$ and refer to the groups which admit an action of $\widetilde{B}_n$ as $\widetilde{B}_n$-groups. The group $\widetilde{B}_n$ is an extension of a solvable group by a symmetric group. We distinguish special elements in $\widetilde{B}_n$-groups which we call prime elements and we give a criterion for an element to be prime. $\widetilde{B}_n$-groups appear as fundamental groups of complements of branch curves. © 1997 Elsevier Science B.V.

Keywords: Braid group

AMS classification: 20F36

0. Introduction

In this paper we describe a quotient of the Artin braid group by commutators of transversal half-twists and investigate its group actions. These groups turned out to be extremely important in describing fundamental groups of complements of branch curves (see, e.g., [5]). The description here is completely independent from the algebraic-geometrical background and provides an algebraic study of the groups involved, using a topological approach to the braid group. We denote the quotient by $\widetilde{B}_n$ and refer to the groups which admit an action of $\widetilde{B}_n$ as $\widetilde{B}_n$-groups. In particular, we study $\widetilde{P}_n$, the image of the pure braid group in $\widetilde{B}_n$, and prove that $\widetilde{B}_n$ is an extension of a solvable group by a symmetric group. The main results on the structure of $\widetilde{B}_n$ are Theorem 6.4

☆ This research was partially supported by the Minerva Foundation from Germany and the Emmy Noether Research Institute.

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PII S0166-8641(96)00154-X

and Corollary 6.5. We distinguish special elements in $\widetilde{B}_n$-groups which we call prime elements, compute the action of half-twists on prime elements (Sections 2–4), and finally we give a criterion for an element to be prime (see Proposition 7.1). This criterion will be applied to the study of fundamental groups of complements of branch curves.

Fundamental groups related to algebraic varieties are very important in classification problems and in topological studies in algebraic geometry. These groups are very difficult to compute. The group $\widetilde{B}_n$ and the $\widetilde{B}_n$-groups appeared when we were computing such groups. It turns out that *all* new examples of fundamental groups of complements of branch curves are $\widetilde{B}_n$-groups and unlike previous expectations they are "almost solvable" (like $\widetilde{B}_n$ itself) . Our future plans are to study $\widetilde{B}_n$-groups indpendent of the fundamental group problems. The ultimate goal is to classify all $\widetilde{B}_n$-groups.

The paper is divided as follows:

Throughout this paper we use the following notations: D is a disc, K is a finite subset of D, $n = \#K$ ($n \geqslant 4$), the braid group $B_n = B_n[D, K]$ is the group of all diffeomorphisms $\beta: D \to D$ which preserve K and act as an identity on ∂D, under the equivalence relation that $\beta_1 \sim \beta_2$ if their actions on $\pi_1(D - K, *)$ coincide. For any elements X, Y in a group G we write $X_Y = Y^{-1}XY$. (The ordinary notation for conjugation is $X^Y = YXY^{-1}$.)

1. Definition of $\widetilde{B}_n$

In this section we define the group $\widetilde{B}_n$. This group and the groups on which it acts (called $\widetilde{B}_n$-groups) are the central objects of our investigation. We also introduce the basic notions of a frame and a good quadrangle. In Claim 1.1 and Lemma 1.2 we establish certain basic identities which will be used repeatedly in later sections.

We recall here the definition of a half-twist in the braid group.

Definition (*Half-twist w.r.t.* $[-1/2, 1/2]$). Consider D_1, the unit disc, $\pm 1/2 \in D_1$. Take $\rho: [0,1] \to [0,1]$ continuous such that $\rho(r) = \pi$ for $r \leqslant 1/2$ and $\rho(1) = 0$. Define $\delta: D_1 \to D_1$ by $\delta(re^{i\theta}) = re^{i(\theta+\rho(r))}$. Clearly, $\delta(1/2) = -1/2$, $\delta(-1/2) = 1/2$, and $\delta|_{\partial D_1} = \mathrm{Id}$. The disc of radius $1/2$ rotates $180°$ counterclockwise. Outside of this disc it rotates by smaller and smaller angles till it is fixed on the unit circle. Thus we get a braid $[\delta] \in B_2[D_1, \{\pm 1/2\}]$. $[\delta]$ is called the half-twist w.r.t. the segment $[-1/2, 1/2]$.

Using the above definition we define a general half-twist.

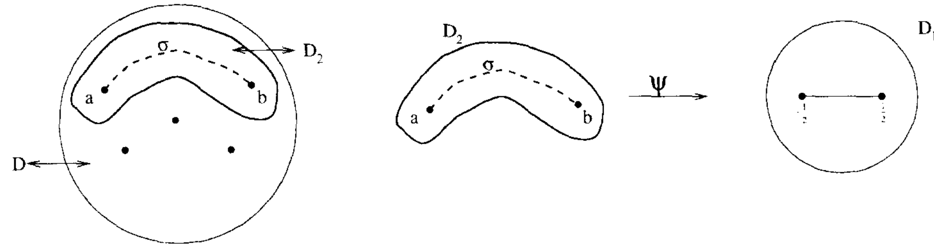


Fig. 1. Half-twist w.r.t. a path.

Definition ($H(\sigma)$, *half-twist w.r.t. a path* σ). Let D, K be as above, $a, b \in K$. Let σ be a path from a to b which does not meet any other point of K. We take a small topological disc $D_2 \subset D$ such that $\sigma \subset D_2$, $D_2 \cap K = \{a, b\}$, see Fig. 1. We take a diffeomorphism $\psi : D_2 \to D_1$ (unit disc) such that $\psi(\sigma) = [-1/2, 1/2]$, $\psi(a) = -1/2$, $\psi(b) = 1/2$. We consider a "rotation" $\psi\delta\psi^{-1} : D_2 \to D_2$; $\psi\delta\psi^{-1}$ is the identity on the boundary of D_2. We extend it to D by the identity. We define the half-twist $H(\sigma)$ to be the conjugacy class of the extension of $\psi\delta\psi^{-1}$.

We sometimes denote the path by x and the corresponding half-twist $H(x)$ by X.

Definition. Let D, K be as denoted above. Let $H(\sigma_1)$ and $H(\sigma_2)$ be two half-twists in $B_n = B_n[D, K]$. We say that $H(\sigma_1)$ and $H(\sigma_2)$ are (see Fig. 2):

(i) *weakly disjoint* if $\sigma_1 \cap \sigma_2 \cap K = \emptyset$.

(ii) *transversal* if σ_1 and σ_2 are weakly disjoint and intersect each other exactly once (and not in any point of K), i.e., $\sigma_1 \cap \sigma_2 =$ one point, $\sigma_1 \cap \sigma_2 \cap K = \emptyset$.

(iii) *disjoint* if $\sigma_1 \cap \sigma_2 = \emptyset$.

(iv) *adjacent* if $\sigma_1 \cap \sigma_2 \cap K =$ one point.

(v) *consecutive* if they are adjacent and $\sigma_1 \cap \sigma_2$ do not intersect outside of K, i.e., $\sigma_1 \cap \sigma_2 =$ point $\in K$.

(vi) *cyclic* if $\sigma_1 \cap \sigma_2 = 2$ points $\in K$.

Claim 1.0. *Let X, Y be two half-twists in B_n. Then:*

(i) *If X, Y are disjoint, then $[X, Y] = 1$, i.e., $XY = YX$.*

(ii) *If X, Y are consecutive, then $\langle X, Y\rangle = XYXY^{-1}X^{-1}Y^{-1} = 1$, thus $XYX = YXY$, $X_{Y^{-1}} = Y_X$ and $X_{Y^{-1}X^{-1}} = Y$. We say then that X and Y satisfy the triple relation.*

(iii) *If $X = H(x)$ is represented by a diffeomorphism β and $Y = H(y)$, then $Y_X = X^{-1}YX = H((y)\beta)$. If X and Y are consecutive, then $(y)\beta$, x and y create a triangle whose interior contains no points of K (see Fig. 3).*

(iv) *If X_1, X_2, X_3 are three consecutive half-twists (see, for example, Fig. 6) then $[X_2, (X_2)_{X_1X_3}] = (X_3)^{-2}_{X_2^{-1}}(X_1)^{-2}_{X_2^{-1}}X_1^2X_3^2$.*

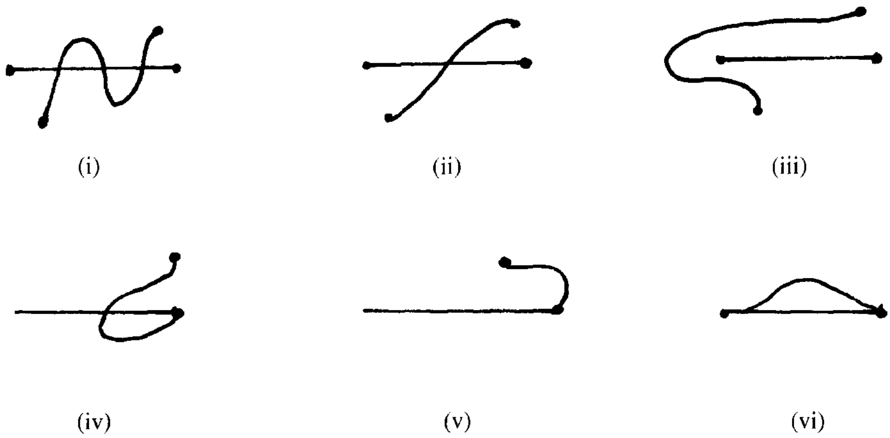


Fig. 2. Relation of half-twists.

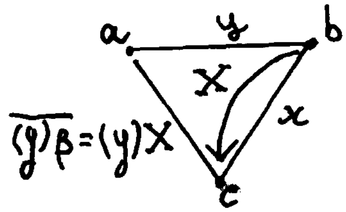


Fig. 3.

Proof. We will use the following observation: if a half-twist in B_n is determined by a 180° rotation around σ, then its inverse is determined by a clockwise rotation. An equation in $B_n[D, K]$ may always be verified by studying the action of each side of the equation on a set of generators of the fundamental group $\pi_1(D - K, *)$. The action of a half-twist on a simple loop is easy to determine: if $a, b \in K$, and Γ_1 and Γ_2 are two simple loops around a and b, respectively, and if σ connects a and b ($\sigma \cap K = \{a, b\}$), then $H(\sigma)$ transfers Γ_1 to Γ_2, Γ_2 to $\Gamma_2\Gamma_1\Gamma_2^{-1}$, and does not move any simple loop around any other point of K. $H(\sigma)^{-1}$ transfers Γ_1 to $\Gamma_1^{-1}\Gamma_2\Gamma_1$ and Γ_2 to Γ_1.

Claims (i), (ii) and (iii) follow from the above observation. For example, for (iii), and X, Y consecutive half-twists with $x \cap y = b$ note that β transfers b to c and fixes a (see Fig. 3). Thus it transfers y to a path connecting a and c. Thus $H((y)\beta)$ fixes b and interchanges a and c. The same is true for $X^{-1}YX$ and thus they coincide. If X and Y are disjoint, (iii) is a trivial result of (i) and the above observation. Items (i) and (ii) are well-known properties of the braid group.

Claim (iv) is a consequence of (i) and (ii) as follows: In B_n $[X_1, X_3] = 1$ and $\langle X_1, X_2\rangle = \langle X_3, X_2\rangle = 1$. Thus, by (ii), $X_iX_j^{-1}X_i^{-1} = X_j^{-1}X_i^{-1}X_j$ for $(i, j) = (2, 3)$, $(1, 2)$, $(3, 2)$, $(2, 1)$. We shall use these relations in the following list of equations:

$$
\begin{aligned}
[X_2, (X_2)_{X_1X_3}] &= [X_2, X_3^{-1}X_1^{-1}X_2X_1X_3] \\
&= X_2X_3^{-1}X_1^{-1}X_2X_1\underbrace{X_3X_2^{-1}X_3^{-1}}X_1^{-1}X_2^{-1}X_1X_3 \\
&= X_2X_3^{-1}X_1^{-1}X_2X_1\overbrace{X_2^{-1}X_3^{-1}X_2}\,\underbrace{X_1^{-1}X_2^{-1}}_{}X_1X_3 \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \ X_1^{-1}X_2^{-1}X_1 \\
&= X_2X_3^{-1}X_1^{-1}X_2\underbrace{X_1X_2^{-1}X_1^{-1}}_{X_2^{-1}X_1^{-1}X_2}X_3^{-1}X_2^{-1}X_1X_1X_3 \\
&= X_2X_3^{-1}X_1^{-2}\underbrace{X_2X_3^{-1}X_2^{-1}}_{X_3^{-1}X_2^{-1}X_3}X_1^2X_3 \\
&= X_2X_3^{-2}X_1^{-2}X_2^{-1}X_1^2X_3^2 \\
&= (X_3)^{-2}_{X_2^{-1}}\,(X_1)^{-2}_{X_2^{-1}}X_1^2X_3^2. \qquad \square
\end{aligned}
$$

Remark. (a) $XYX = YXY$ is called the *triple relation.*

(b) Claim (iv) will be used in the proof of Lemma 1.2 which is the first step in understanding $\widetilde{B}_n$.

Definition ($\widetilde{B}_n$). Let $n \geqslant 4$. $\widetilde{B}_n$ is the quotient of B_n, the braid group of order n, by the subgroup normally generated by the commutators $[H(\sigma_1), H(\sigma_2)]$, where $H(\sigma_1)$ and $H(\sigma_2)$ are transversal half-twists.

Notation. Let $Y \in B_n$. We denote the image of Y in $\widetilde{B}_n$ by $\widetilde{Y}$. If Y is a half-twist in B_n we call $\widetilde{Y}$ a half-twist in $\widetilde{B}_n$. We call two half-twists $\widetilde{Y}$, $\widetilde{X}$ in $\widetilde{B}_n$ disjoint (or weakly disjoint, adjacent, consecutive, transversal) if Y, X are disjoint (or weakly disjoint, adjacent, consecutive, transversal).

Definition (*A frame of* $\widetilde{B}_n$). Let $K = \{a_1, \ldots, a_n\}$. Let σ_i, $i = 1, \ldots, n-1$, be a consecutive sequence of simple paths in D connecting the points of K such that $\sigma_i \cap K = \{a_i, a_{i+1}\}$ and

$$
\sigma_i \cap \sigma_j = \begin{cases} \emptyset, & i \neq j, j+1, \\ a_j, & i = j-1, \\ a_{j+1}, & i = j+1. \end{cases}
$$

Then $\{X_i = H(\sigma_i)\}$ is a frame of B_n and $\{\widetilde{X}_i\}$ is a frame of $\widetilde{B}_n$.

Remark. We also refer to a frame as a standard base of $\widetilde{B}_n$. By the classical Artin theorem, a frame generates $\widetilde{B}_n$ (see [1]).

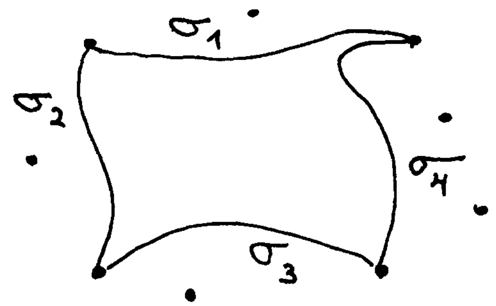


Fig. 4. Good quadrangle.

Definition (*Polarized half-twist, polarization*). We say that a half-twist $X \in B_n$ (or $\widetilde{X}$ in $\widetilde{B}_n$) is polarized if we choose an order on the end points of X. The order is called the polarization of X (or $\widetilde{X}$).

Definition (*Orderly adjacent*). Let X, Y (or $\widetilde{X}$, $\widetilde{Y}$) be two adjacent polarized half-twists in B_n (respectively in $\widetilde{B}_n$). We say that X, Y (or $\widetilde{X}$, $\widetilde{Y}$) are *orderly adjacent* if their common point is the "end" of one of them and the "origin" of another.

Definition (*Good quadrangle*). Let $H(\sigma_i)$, $i = 1, \ldots, 4$, be four half-twists such that $H(\sigma_i)$ is consecutive to $H(\sigma_{i+1})$ (mod 4), $H(\sigma_i)$ is disjoint from $H(\sigma_{i+2})$ (mod 4) and in the disc with boundary $\bigcup_{i=1}^{4} \sigma_i$ there are no points of K (see Fig. 4).

We say that $\{H(\sigma_i)\}$ is a good quadrangle in B_n, and $\{\widetilde{H}(\sigma_i)\}$ is a good quadrangle in $\widetilde{B}_n$.

Claim 1.1. (a) *Transversal, disjoint $\Rightarrow$ weakly disjoint. Consecutive $\Rightarrow$ adjacent.*

(b) *Every two transversal or disjoint half-twists commute. Every two consecutive half-twists satisfy the triple relation ($\widetilde{X}\widetilde{Y}\widetilde{X} = \widetilde{Y}\widetilde{X}\widetilde{Y}$ or $\widetilde{X}^{-1}\widetilde{Y}\widetilde{X} = \widetilde{Y}\widetilde{X}\widetilde{Y}^{-1}$).*

(c) *Any two half-twists are conjugate to each other. It is possible to choose the conjugacy element such that it commutes with any half-twist which is disjoint from the given two half-twists.*

(d) *Any two pairs of disjoint (or transversal, consecutive, cyclic) half-twists are conjugate to each other.*

(e) *Any two good quadrangles are conjugate to each other.*

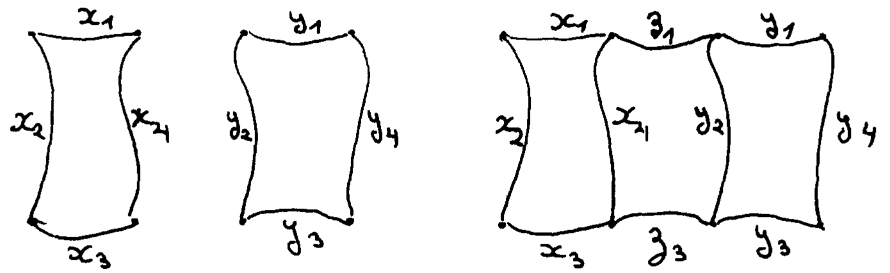
(f) *Every two pairs of orderly adjacent (nonorderly adjacent) consecutive half-twists are conjugate to each other, preserving the polarization.*

Proof. If the statements hold for B_n, they also hold for $\widetilde{B}_n$. We shall prove them for B_n.

(a) By simple geometric observation in (D, k).

(b) By Claim 1.0(i)–(ii).

(c) We consider here only two cases: the half-twists X, Y are consecutive; the half-twists X, Y are disjoint. For X and Y consecutive we get from (b) that $(X)_{YX} = Y$. For



(a) (b)

Fig. 5.

$X = H(x)$ and $Y = H(y)$ disjoint half-twists, we choose a simple path σ connecting an end point of x with an end point of y and do not meet any other point of K and any other point of $x \cup y$. Denote $Z = H(\sigma)$. Since X is consecutive to Z and Z is consecutive to Y we get from the first case that $(X)_{ZXYZ} = (X_{ZX})_{YZ} = Z_{YZ} = Y$.

(d) We only consider here the case in which (X_1, X_2) and (Y_1, Y_2) are two pairs of disjoint half-twists. By (c) there exist b_1 such that $(X_1)_{b_1} = Y_1$ and b_2 such that $(X_2)_{b_2} = Y_2$. Since X_1 is disjoint from X_2, for b_1 chosen as in (c) we get $(X_2)_{b_1} = X_2$. Similarly, $(Y_1)_{b_2} = Y_1$. Let $b = b_1 b_2$. We get $(X_1)_b = Y_1$ and $(X_2)_b = Y_2$.

(e) Without loss of generality we can assume that the quadrangles are disjoint (the paths and the interior). Take two disjoint quadrangles as in Fig. 5(a). Let z_i $(i = 1, 3)$ be a simple path connecting the end point of x_i with the beginning point of y_i $(i = 1, 3)$, see Fig. 5(b). Let $b_1 = Z_1 X_1 Y_1 Z_1$, $b_3 = Z_3^{-1} X_3^{-1} Y_3^{-1} Z_3^{-1}$. The choice of b_1 assures that $(X_1)_{b_1} = Y_1$, $(X_3)_{b_3} = Y_3$ (see (c)). Moreover, $(X_3)_{b_1} = X_3$, $(Y_1)_{b_3} = Y_1$. Let $b = b_1 b_3$. Clearly, $(X_1)_b = Y_1$ and $(X_3)_b = Y_3$. We still have to show that $(X_2)_b = Y_2$, $(X_4)_b = Y_4$. Now, $(X_2)_b = (X_2)_{Z_1 X_1 Y_1 Z_1 Z_3^{-1} X_3^{-1} Y_3^{-1} Z_3^{-1}}$. By Claim 1.0(i) $(X_2)_{Z_1} = X_2$. By Claim 1.0(iii) $(X_2)_{X_1} = H(t_1)$ where t_1 connects the two end points of X_2 and X_1 that do not intersect (see Fig. 5(b)). By Claim 1.0(i), $H(t_1)_{Y_1} = H(t_1)$. By Claim 1.0(iii), $H(t_1)_{Z_1} = H(t_2)$. Alternately, we apply Claim 1.0(i) and Claim 1.0(iii) and finally get $(X_2)_b = Y_2$. Similarly, we get $(X_4)_b = Y_4$.

(f) The choices made in (b) preserve the polarization. □

Lemma 1.2(b) is an important relation in $\widetilde{B}_n$; its discovery was the first step in understanding the structure of $\widetilde{B}_n$.

Lemma 1.2. *If* $\{Y_i\}$, $i = 1, \ldots, 4$, *is a good quadrangle in* B_n, *then*

(a) $\widetilde{Y}_1 \widetilde{Y}_3 = \widetilde{Y}_3 \widetilde{Y}_1$,

(b) $Y_1^2 Y_3^2 Y_4^{-2} Y_2^{-2}$ *normally generates* $\ker(B_n \to \widetilde{B}_n)$. *In particular,* $\widetilde{Y}_1^2 \widetilde{Y}_3^2 = \widetilde{Y}_2^2\, \widetilde{Y}_4^2$.

Proof. (a) Since $\widetilde{Y}_1$ and $\widetilde{Y}_3$ are disjoint half-twists, Claim 1.1(d) implies that they commute.

(b) Since all pairs of transversal half-twists are conjugate to each other, $\ker(B_n \to \widetilde{B}_n)$ is normally generated by any such commutator or a conjugate of it. So to prove (b) it

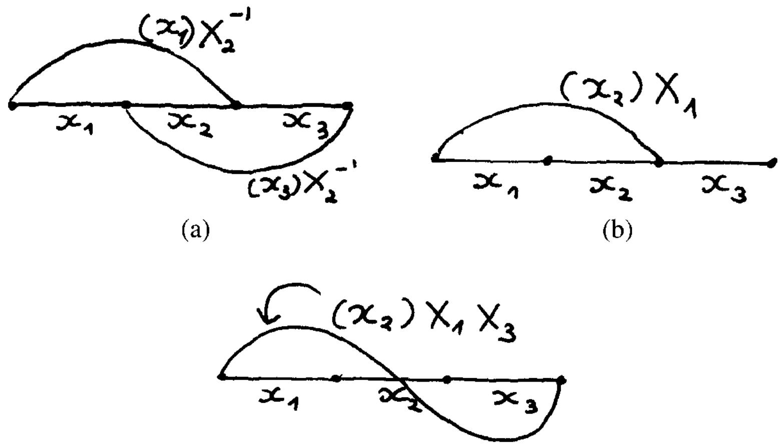


Fig. 6.

is enough to show that $Y_1^2Y_3^2Y_4^{-2}Y_2^{-2}$ is conjugate to a commutator of transversal half-twists.

Let X_1, X_2, X_3 be 3 half-twists such that X_1 and X_2 are consecutive, X_2 and X_3 are consecutive and X_1 and X_3 are disjoint. Denote $X_i = H(x_i)$, $i = 1,2,3$. We use Claim 1.0(iii) to get a geometric description of

$$(X_3)_{X_2^{-1}} = H\big((x_3)X_2^{-1}\big)\ (= (X_2)_{X_3})$$

and of

$$(X_1)_{X_2^{-1}} = H\big((x_1)X_2^{-1}\big)\ (= (X_2)_{X_1})$$

(see Fig. 6(a)). We thus get that $X_1, (X_3)_{X_2^{-1}}, X_3, (X_1)_{X_2^{-1}}$ is a good quadrangle.

By Claim 1.0(iv),

$$\big(X_3^{-2}\big)_{X_2^{-1}}(X_1)_{X_2^{-1}}^{-2}X_1^2X_3^2 = \big[X_2, (X_2)_{X_1X_3}\big].$$

By Claim 1.0(iii) applied twice, $(X_2)_{X_1X_3}$ is transversal to X_2 (see Figs. 6(b) and 6(c)). Therefore $(X_3)_{X_2^{-1}}^{-1}(X_1)_{X_2^{-1}}^{-2}X_1^2X_3^2$ is a commutator of transversal half-twists.

Since every two quadrangles are conjugate to each other, $Y_2^{-2}Y_4^{-2}Y_1^2Y_3^2$ is conjugate to a commutator of transversal half-twists. As explained in the beginning this is enough to conclude that $Y_1^2Y_3^2Y_4^{-2}Y_2^{-2}$ normally generates $\ker(B_n \to \widetilde{B}_n)$. In particular, $\widetilde{Y}_1^2\widetilde{Y}_3^2\widetilde{Y}_4^{-2}\widetilde{Y}_2^{-2} = 1$ and $\widetilde{Y}_1^2\widetilde{Y}_3^2 = \widetilde{Y}_2^2\widetilde{Y}_4^2$. □

2. $\widetilde{B}_n$-groups and prime elements

In this section we define a $\widetilde{B}_n$-group, prime elements of $\widetilde{B}_n$-groups and their supporting half-twist. We present basic properties of a prime element (Lemma 2.1) and we prove two criteria for an element to be prime (Lemmas 2.2 and 2.4). These criteria are necessary steps for Proposition 7.1 in which we prove a criterion for prime elements which is easier to apply. These elements are called prime elements since they satisfy an existence and a uniqueness property as proven in Theorem 3.3.

Definition (*$\widetilde{B}_n$-group*). A group G is called a $\widetilde{B}_n$-group if there exists a homomorphism $\widetilde{B}_n \to \mathrm{Aut}(G)$. For $g \in G$ and $b \in \widetilde{B}_n$ we denote by g_b or $(g)_b$, the element obtained from b acting on g.

Definition (*Prime element, supporting half-twist, corresponding central element, axioms of prime elements*). Let G be a $\widetilde{B}_n$-group. An element $g \in G$ is called a prime element of G if there exists a half-twist $X \in B_n$ and $\tau \in \mathrm{Center}(G)$ with $\tau^2 = 1$ and $\tau_b = \tau\ \forall b \in \widetilde{B}_n$ such that:

(1) $g_{\widetilde{X}^{-1}} = g^{-1}\tau$;

(2) for every half-twist Y adjacent to X we have:

$$\text{(a)}\quad g_{\widetilde{X}\widetilde{Y}^{-1}\widetilde{X}^{-1}} = g_{\widetilde{X}}^{-1} g_{\widetilde{X}\widetilde{Y}^{-1}},$$

$$\text{(b)}\quad g_{\widetilde{Y}^{-1}\widetilde{X}^{-1}} = g^{-1} g_{\widetilde{Y}^{-1}};$$

(3) for every half-twist Z disjoint from X, $g_{\widetilde{Z}} = g$.

The half-twist X (or $\widetilde{X}$) is called the *supporting half-twist* of g.

The element τ is called the *corresponding central element*.

We shall refer to (1), (2) and (3) as *Axiom* (1), *Axiom* (2), *Axiom* (3), respectively.

The following lemma gives the basic properties of prime elements.

Lemma 2.1. *Let G be a $\widetilde{B}_n$-group. Let g be a prime element in G with supporting half-twist X and corresponding central element τ. Then:*

(1) $g_{\widetilde{X}} = g_{\widetilde{X}^{-1}} = g^{-1}\tau$, $g_{\widetilde{X}^2} = g$.

(2) $g_{\widetilde{Y}^{-2}} = g\tau$ *$\forall Y$ consecutive half-twist to X.*

(3) $[g, g_{\widetilde{Y}^{-1}}] = \tau$ *$\forall Y$ consecutive half-twist to X.*

Proof. (1)

$$g_{\widetilde{X}^{-2}} = (g_{\widetilde{X}^{-1}})_{\widetilde{X}^{-1}} = (g^{-1}\tau)_{\widetilde{X}^{-1}} = (g^{-1})_{\widetilde{X}^{-1}} \cdot \tau = (g^{-1}\tau)^{-1}\tau = g$$
$$\Rightarrow g_{\widetilde{X}} = g_{\widetilde{X}^{-1}} \overset{\text{Axiom (1)}}{=} g^{-1}\tau.$$

(2)

$$g_{\widetilde{X}^{-1}\widetilde{Y}^{-1}\widetilde{X}^{-1}} = (g_{\widetilde{X}^{-1}})_{\widetilde{Y}^{-1}\widetilde{X}^{-1}} = (g^{-1}\tau)_{\widetilde{Y}^{-1}\widetilde{X}^{-1}} = (g_{\widetilde{Y}^{-1}\widetilde{X}^{-1}})^{-1} \cdot \tau$$
$$\overset{\text{Axiom (2)}}{=} g_{\widetilde{Y}^{-1}}^{-1} \cdot g \cdot \tau.$$

On the other hand,

$$g_{\widetilde{X}^{-1}\widetilde{Y}^{-1}\widetilde{X}^{-1}} = g_{\widetilde{Y}^{-1}\widetilde{X}^{-1}\widetilde{Y}^{-1}} = (g_{\widetilde{Y}^{-1}\widetilde{X}^{-1}})_{\widetilde{Y}^{-1}} \overset{\text{Axiom (2)}}{=} (g^{-1}g_{\widetilde{Y}^{-1}})_{\widetilde{Y}^{-1}}$$
$$= g^{-1}_{\widetilde{Y}^{-1}} \cdot g_{\widetilde{Y}^{-2}}.$$

Thus, $g_{\widetilde{Y}^{-2}} = g\tau$.

(3)

$$g_{\widetilde{X}\widetilde{Y}^{-1}X^{-1}} \overset{\text{Axiom (2)}}{=} g_{\widetilde{X}}^{-1} \cdot g_{\widetilde{X}\widetilde{Y}^{-1}} \overset{(1)}{=} g_X^{-1} g_{\widetilde{X}\widetilde{Y}^{-1}} = g \cdot g^{-1}_{\widetilde{Y}^{-1}} \cdot \tau^2.$$

On the other hand,

$$g_{\widetilde{X}\widetilde{Y}^{-1}\widetilde{X}^{-1}} \overset{(1)}{=} (g^{-1}\tau)_{\widetilde{Y}^{-1}\widetilde{X}^{-1}} \overset{\text{Axiom (2)}}{=} \left(g^{-1} \cdot g_{\widetilde{Y}^{-1}}\right)^{-1} \cdot \tau = g^{-1}_{\widetilde{Y}^{-1}} \cdot g \cdot \tau.$$

Thus, $g \cdot g^{-1}_{\widetilde{Y}^{-1}} \cdot = g^{-1}_{\widetilde{Y}^{-1}} \cdot g\tau$, $g_{\widetilde{Y}^{-1}} \cdot g \cdot g^{-1}_{\widetilde{Y}^{-1}} \cdot g^{-1} = \tau^{-1} = \tau$, and $[g_{\widetilde{Y}^{-1}}, g] = \tau$. □

Lemma 2.2. *Let G be a $\widetilde{B}_n$-group. Let g be a prime element in G with supporting half-twist X and corresponding central element τ. Let $b \in \widetilde{B}_n$. Then g_b is a prime element with supporting half-twist X_b and central element τ.*

Proof. We use the fact that $(a_b)_c = (a_c)_{b_c}$ and $(ab)_c = a_c b_c$. We have to prove three properties:

(1)

$$g_{\widetilde{X}^{-1}} = g^{-1}\tau \Rightarrow (g_{\widetilde{X}^{-1}})_b = (g^{-1}\tau)_b \Rightarrow g_{\widetilde{X}^{-1}{}_b} = g_b^{-1}\tau \Rightarrow g_{bb^{-1}\widetilde{X}^{-1}b} = (g_b)^{-1}\tau$$
$$\Rightarrow (g_b)_{\widetilde{X}_b^{-1}} = (g_b)^{-1} \cdot \tau.$$

(2) Let Y be a half-twist adjacent to X_b. Then $Y_{b^{-1}}$ is adjacent to X and satisfies Axiom (2) of prime elements for g, X and Y_b^{-1}. Namely:

$$g_{\widetilde{Y}^{-1}_{b^{-1}}\widetilde{X}^{-1}} = g^{-1}g_{\widetilde{Y}^{-1}_{b^{-1}}} \quad \text{and} \quad g_{\widetilde{X}\widetilde{Y}^{-1}_{b^{-1}}\widetilde{X}^{-1}} = g_{\widetilde{X}}^{-1} g_{\widetilde{X}\widetilde{Y}^{-1}_{b^{-1}}}.$$

(a) $$g_{\widetilde{Y}^{-1}_{b^{-1}}\widetilde{X}^{-1}} = g^{-1}g_{\widetilde{Y}^{-1}_{b^{-1}}} \Rightarrow \left(g_{\widetilde{T}^{-1}_{b^{-1}}\widetilde{X}^{-1}}\right)_b = \left(g^{-1}g_{\widetilde{Y}^{-1}_{b^{-1}}}\right)_b$$
$$\Rightarrow (g_b)_{(\widetilde{Y}_{b^{-1}}\widetilde{X}^{-1})_b} = (g_b)^{-1}(g_b)_{\widetilde{Y}^{-1}} \Rightarrow (g_b)_{\widetilde{Y}\widetilde{X}_b^{-1}} = g_b^{-1} \cdot (g_b)_{\widetilde{Y}^{-1}}.$$

(b) $$g_{\widetilde{X}\widetilde{Y}^{-1}_{b^{-1}}\widetilde{X}^{-1}} = g_{\widetilde{X}}^{-1} g_{\widetilde{X}\widetilde{Y}^{-1}_{b^{-1}}} \Rightarrow (g_b)_{\widetilde{X}_b\widetilde{Y}^{-1}\widetilde{X}_b^{-1}} = (g_b^{-1})_{\widetilde{X}_b}(g_b)_{\widetilde{X}_b\widetilde{Y}^{-1}}.$$

(3) Let Z be a half-twist disjoint from X_b. Then $Z_{b^{-1}}$ is disjoint from X. Then $g_{\widetilde{Z}_{b^{-1}}} = g$. We conjugate $g_{\widetilde{Z}_b^{-1}} = g$ by b to get: $(g_b)_{\widetilde{Z}} = g_b$. □

We need the following technical lemma on B_n to prove later a criterion for a prime element in a $\widetilde{B}_n$-group.

Claim 2.3. *Let $(X_1, \ldots, X_{n-1})$ be a frame in $B_n = B_n[D, K]$. Let*

$$C(X_1) = \left\{b \in B_n \,\middle|\, [b, X_1] = 1\right\} \quad (\textit{centralizer of } X_1),$$
$$C_p(X_1) = \left\{b \in B_n \,\middle|\, (X_1)_b = X_1, \textit{ preserving the polarization}\right\}.$$

Let $\sigma = X_2X_1^2X_2$. Then $C(X_1)$ is generated by $\{X_1, \sigma, X_3, \ldots, X_n\}$, $C_p(X_1)$ is generated by $\{X_1^2, \sigma, X_3, \ldots, X_n\}$.

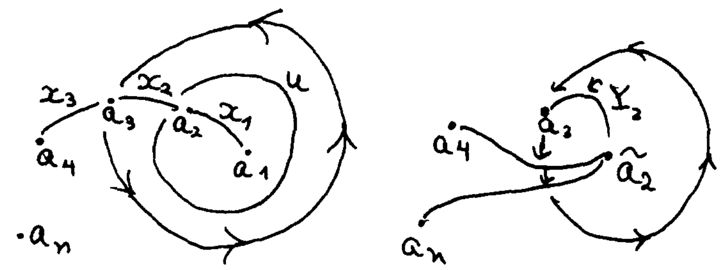


Fig. 7.

Proof. Let $K = \{a_1, \ldots, a_n\}$. Let $x_1, \ldots, x_{n-1}$ be a system of consecutive simple paths in D, such that $X_i = H(x_i)$ ($H(x_i)$ is the half-twist corresponding to x_i; x_i connects a_i with a_{i+1}). Let $\Gamma_1, \ldots, \Gamma_n$ be a free geometric base of $\pi_1(D - K, *)$ consistent with $(X_1, \ldots, X_{n-1})$ (that is, $(\Gamma_{i+1})X_i = \Gamma_i$, $(\Gamma_i)X_i = \Gamma_i \Gamma_{i+1} \Gamma_i^{-1}$, $(\Gamma_j)X_i = \Gamma_j$ for $j \neq i, i+1$). We can assume that the x_i do not intersect the "tails" of $\Gamma_1, \ldots, \Gamma_n$.

Let K_1 be a finite set of D obtained from $K \cup \{x_1\}$ by contracting x_1 to a point $\tilde{a}_2 \in x_1$, $K_1 = \{\tilde{a}_2, a_3, \ldots, a_n\}$. Let $B_{n-1} = B_{n-1}[D, K_1]$. Let $Y_2, \ldots, Y_{n-1}$ be a frame of B_{n-1}, where Y_i can be identified with X_i for $i \geqslant 3$.

Let $H = \{b \in B_{n-1} \mid (\tilde{a}_2)b = \tilde{a}_2\}$. From the short exact sequence

$$1 \to P_{n-1} \hookrightarrow B_{n-1} \to S_{n-1} \to 1$$

(P_{n-1} the pure braid group) we can conclude that H is generated by $Y_3, \ldots, Y_{n-1}$ and by the generators of P_{n-1}. We remove the generators of P_{n-1} that can be expressed in terms of $Y_3, \ldots, Y_{n-1}$ (see [1,2] and [3, Section IV]) and conclude that H is generated by $Y_2^2, Y_3, \ldots, Y_n$. The element Y_2^2 corresponds to the motion $\mathcal{M}'$ of $\tilde{a}_2, a_3, \ldots, a_n$ described as follows: $\tilde{a}_2, a_4, \ldots, a_n$ stays in place and a_3 is moving around $\tilde{a}_2$ in the positive direction (see Figs. 7(a) and (b)).

We define a homomorphism $\Phi : C_p(X_1) \to H$ as follows:

Let U be a "narrow" neighborhood of x_1 such that $\lambda = \partial U$ is a simple loop. Take $b \in C_p(X_1)$. It can be represented by a diffeomorphism $\beta : D \to D$ such that $\beta(K) = K$, $\beta|_{\partial D} = \mathrm{Id}_{\partial D}$ and $\beta|_{\overline{U}} = \mathrm{Id}_{\overline{U}}$ ($\overline{U} = U \cup \lambda$).

The diffeomorphism β also defines an element of $B_{n-1}[D, K_1]$. This element is in fact in H since $\tilde{a}_2 \in x_1$ and thus $(\tilde{a}_2)\beta = \tilde{a}_2$. Denote this element by $\Phi(b)$. The map Φ constructed in this way is obviously a homomorphism, $\Phi : C_p(x_1) \to H$. Clearly, $X_3, \ldots, X_{n-1} \in C_p(X_1)$. Clearly, $\Phi(X_i) = Y_i$ for $i \geqslant 3$. Let $\mathcal{M}$ be the following motion (D, K): a_1, a_2, a_4, $\ldots$, a_n are stationary and a_3 goes around a_1, a_2 in the positive direction (Fig. 7(b)). Let u be the braid in $C_p(X_1)$ induced from the motion $\mathcal{M}$. Clearly, $\Phi(u) = Y_2^2$. Thus, Φ is onto and $\Phi(u)$, $\Phi(X_3)$, $\ldots$, $\Phi(X_{n-1})$ generate H. One can check that $u = Z_{31}^2 Z_{32}^2$. But $Z_{31} = X_2 X_1 X_2^{-1}$ and $Z_{32} = X_2$. Thus $u = \sigma$. Thus $C_p(X_1)$ is generated by σ, X_3, $\ldots$, X_{n-1} and a set of generators for $\ker \Phi$.

Consider $\pi_1(D-K\cup x_1,*)$. Let $\widetilde{\Gamma}_2$ be the path obtained by connecting λ with $*\in\partial D$ by a simple path intersecting each of $\Gamma_3,\ldots,\Gamma_n$ only at $*$. We get a (free) geometric base $\widetilde{\Gamma}_2$, Γ_3, $\ldots$, Γ_n of $\pi_1(D-(K\cup x_1),*)$. It is obvious that $\widetilde{\Gamma}_2=\Gamma_1\Gamma_2$. $\Phi(b)$ defines in a natural way an automorphism of $\pi_1(D_k\cup\{X_1\},*)$ such that $\Phi(b)$ does not change the product $\widetilde{\Gamma}_2\Gamma_3\cdots\Gamma_n$, and $(\widetilde{\Gamma}_2)\Phi(b)$ is a conjugate of $\widetilde{\Gamma}_2$.

Consider now any $Z\in\ker\Phi$. We have $(\widetilde{\Gamma}_2)Z=\widetilde{\Gamma}_2(\widetilde{\Gamma}_2=\Gamma_1\Gamma_2)$, $(\Gamma_j)Z=\Gamma_j$ $\forall j=3,\ldots,n$. This implies that Z can be represented by a diffeomorphism which is the identity outside of U, that is, $Z=X_1^\ell$, $\ell\in\mathbb{Z}$. Since $Z\in C_p(X_1)$, ℓ must equal 0 (mod 2).

Thus, $C_p(X_1)$ is generated by X_1^2, σ, X_3, $\ldots$, X_{n-1}. Clearly, $C(X_1)$ is generated by $C_p(X_1)$ and X_1. □

Lemma 2.4. *Let $\{X_1,\ldots,X_{n-1}\}$ be a frame in B_n, $(\widetilde{X}_1,\ldots,\widetilde{X}_{n-1})$ their images in $\widetilde{B}_n$. Let G be a $\widetilde{B}_n$-group. Let $u\in G,\tau\in G$ be such that*

(1) $u_{\widetilde{X}_1^{-1}}=u^{-1}\tau$ *with* $\tau^2=1$, $\tau\in\mathrm{Center}(G)$, $\tau_b=\tau\,\forall b\in\widetilde{B}_n$;

(2a) $u_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}=u^{-1}u_{\widetilde{X}_2^{-1}}$;

(2b) $u_{\widetilde{X}_1\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}=u_{\widetilde{X}_1}^{-1}u_{\widetilde{X}_1\widetilde{X}_2^{-1}}$;

(3) $u_{\widetilde{X}_j}=u\,\forall j=3,\ldots,n-1$.

Then u is a prime element in G, and X_1 is its supporting half-twist and τ is its central element.

Proof. Let $Z\in B_n$ be any half-twist disjoint from X_1, $\widetilde{Z}$ be the image of Z in $\widetilde{B}_n$. $\exists b\in B_n$ such that $(X_1)_b=X_1$, $(X_3)_b=Z$. By Claim 2.3, b belongs to the subgroup of B_n generated by $X_1,X_3,\ldots,X_{n-1}$ and $\sigma=X_2X_1^2X_2$. Let $\tilde{b}$ and $\tilde{\sigma}$ be the images of b and σ in $\widetilde{B}_n$. We have

$$\begin{aligned}u_{\tilde{\sigma}^{-1}}&=u_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-2}\widetilde{X}_2^{-1}}=\left(u_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}\right)_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}}=\left(u^{-1}u_{\widetilde{X}_2^{-1}}\right)_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}}\\&=(\tau u)_{\widetilde{X}_2^{-1}}\cdot u_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}}=\tau u_{\widetilde{X}_2^{-1}}\cdot u_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}\\&=\tau u_{\widetilde{X}_2^{-1}}\cdot(\tau u^{-1})_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}=\tau^2u_{\widetilde{X}_2^{-1}}u_{\widetilde{X}_2^{-1}}^{-1}\tau(\tau u)=u.\end{aligned}$$

Then $u_{\tilde{\sigma}}=u$. Now: $u_{\widetilde{X}_j}=u$ for $j\geqslant 3$ (by assumption (3)) and $u_{\widetilde{X}_1^2}=u$ (by assumption (1)). Thus, if X_1 appears in b an even number of times, then $u_b=u$. Otherwise we replace b by bX_1. The "new" b satisfies the same requirement for b as above, as well as the equation $u_{\tilde{b}}=u$. Thus, we can assume $u_{\tilde{b}}=u$. We have

$$u_{\widetilde{Z}}=u_{\tilde{b}^{-1}\widetilde{X}_3\tilde{b}}=u_{\widetilde{X}_3\tilde{b}}=u_{\tilde{b}}=u.$$

Let Y be a half-twist in B_n adjacent to X_1. $\exists b_1\in B_n$ such that $(X_1)_{b_1}=X_1$, $(X_2)_{b_1}=Y$. Let $\tilde{b}_1$ and $\widetilde{Y}$ be the images of b_1 and Y in $\widetilde{B}_n$. As above, we can choose b_1 so that $u_{\tilde{b}_1}=u$. Applying $\tilde{b}_1$ on the assumptions (2a) and (2b) we get (using $u_{\tilde{b}_1}=u$, $(\widetilde{X}_1)_{\tilde{b}_1}=\widetilde{X}_1,(\widetilde{X}_2)_{\tilde{b}_1}=\widetilde{Y}$):

$$u_{\widetilde{Y}^{-1}\widetilde{X}_1^{-1}}=u^{-1}u_{\widetilde{Y}^{-1}}\quad\text{and}\quad u_{\widetilde{X}_1\widetilde{Y}^{-1}\widetilde{X}_1^{-1}}=u_{\widetilde{X}_1}^{-1}u_{\widetilde{X}_1\widetilde{Y}^{-1}}.\qquad\square$$

3. Polarized pairs and uniqueness of coherent pairs

In this section we extend the notion of a polarized half-twist to prime elements, and we create polarized pairs which consist of a prime element and its supporting half-twist when considered polarized. We also extend the notion of braids preserving the polarization to coherent pairs.

The main results of this section are Theorem 3.3 and Corollary 3.5. In Theorem 3.3 we establish the unique existence of a polarized pair, with a given supporting half-twist coherent to an original polarized pair. We denote by $L_{h,\widetilde{X}}(\widetilde{T})$ the unique prime element with supporting half-twist $\widetilde{T}$ in the new polarized pair which is coherent to $(h, \widetilde{X})$. In Corollary 3.5 we prove simultaneous conjugation.

Definition (*Polarized pair*). Let G be a $\widetilde{B}_n$-group, h a prime element of G, X its supporting half-twist. If X is polarized, we say that (h, X) (or $(h, \widetilde{X})$) is a polarized pair with central element τ, $\tau = hh_{\widetilde{X}^{-1}}$.

Definition (*Coherent pairs, anticoherent pairs*). We say that two polarized pairs $(h_1, \widetilde{X}_1)$ and $(h_2, \widetilde{X}_2)$ are coherent (anticoherent) if $\exists \tilde{b} \in \widetilde{B}_n$ such that $(h_1)_{\tilde{b}} = h_2$, $(\widetilde{X}_1)_b = \widetilde{X}_2$, and b preserves (reverses) the polarization.

Corollary 3.1. *Coherent and anticoherent polarized pairs have the same central element.*

Proof. The prime elements of coherent and anticoherent pairs are conjugate to each other. Thus by Lemma 2.2 we get the corollary. □

We need the following lemma to prove later the unique existence of a polarized pair with given supporting half-twist coherent to an original polarized pair.

Lemma 3.2. *Let G be a $\widetilde{B}_n$-group. Let $h \in G$ be a prime element with supporting half-twist X. Let $b \in B_n$. Then $X_b = X$ preserving the polarization $\Rightarrow h_{\tilde{b}} = h$.*

Proof. We can choose a set of standard generators for $B_n[D, K]$, $\{X_1, \ldots, X_{n-1}\}$ with $X_1 = X$. Let $\sigma = X_2 X_1^2 X_2$. Consider $C_p(X_1)$, the centralizer of X_1, preserving the polarization. By Lemma 2.3, $C_p(X_1)$ is the subgroup of B_n generated by X_1^2, σ, X_3, $\ldots$, X_{n-1}. Since $\widetilde{X}_3$, $\ldots$, $\widetilde{X}_{n-1}$ are disjoint from $\widetilde{X}_1$, they do not change h (by Axiom (3) of prime elements). By Lemma 2.1, $h_{\widetilde{X}_1^2} = h$. Consider $h_{\tilde{\sigma}^{-1}} = h_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-2}\widetilde{X}_2^{-1}}$. We have:

$$\begin{aligned} h_{\tilde{\sigma}^{-1}} &= h_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-2}\widetilde{X}_2^{-1}} = \left(h_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}\right)_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}} \overset{\text{Axiom (2)}}{=} \left(h^{-1} h_{\widetilde{X}_2^{-1}}\right)_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}} \\ &= \left(h^{-1}_{\widetilde{X}_1^{-1}} h_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}\right)_{\widetilde{X}_2^{-1}} \overset{\text{Axiom (2)}}{=} \left(\tau h \cdot h^{-1} h_{\widetilde{X}_2^{-1}}\right)_{\widetilde{X}_2^{-1}} \\ &= \tau h_{\widetilde{X}_2^{-2}} \overset{\text{Lemma 2.1(2)}}{=} \tau h \tau = h. \end{aligned}$$

Thus $h_{\tilde{\sigma}} = h$. Thus, for every generator g of $C_p(X)$, $h_{\tilde{g}} = h$. Since $b \in C_p(X)$, $h_{\tilde{b}} = h$. □

Theorem 3.3 (Existence and uniqueness). *Let G be a $\widetilde{B}_n$-group. Let $(h, \widetilde{X})$ be a polarized pair of G. Let $\widetilde{T}$ be a polarized half-twist in $\widetilde{B}_n$. Then there exists a unique prime element $g \in G$ such that $(g, \widetilde{T})$ and $(h, \widetilde{X})$ are coherent.*

Proof. Let $X, T \in B_n$ be polarized half-twists representing $\widetilde{X}$ and $\widetilde{T}$. $\exists b \in B_n$ such that $T = X_b$ preserving the polarization. Let $\tilde{b}$ be the image of b in $\widetilde{B}_n$. Taking $g = h_{\tilde{b}}$ we obtain a polarized pair $(g, \widetilde{T})$ coherent with $(h, \widetilde{X})$. To prove the *uniqueness* of g, assume that $(g_1, \widetilde{T})$ is another polarized pair coherent with $(h, \widetilde{X})$. Then $\exists b_1 \in B_n$ with $g_1 = h_{\tilde{b}_1}$ and $T = X_{b_1}$, preserving the polarization. We have $T = X_{b_1} = X_b$ and $X_{b_1 b^{-1}} = X$. Denote $b_2 = b_1 b^{-1}$, so $X_{b_2} = X$ (preserving the polarization). By the previous lemma, $h_{\tilde{b}_2} = h$. Thus, $h_{\tilde{b}_1} = h_{\tilde{b}}$ or $g = g_1$. □

Definition ($L_{(h,\widetilde{X})}(\widetilde{T})$). Let $(h, \widetilde{X})$ be a polarized pair. Let $\widetilde{T} \in \widetilde{B}_n$. We denote by $L_{(h,\widetilde{X})}(\widetilde{T})$ the unique prime element such that $(L_{(h,\widetilde{X})}(\widetilde{T}), \widetilde{T})$ is coherent with $(h, \widetilde{X})$.

Lemma 3.4. *Assume $(h, \widetilde{X})$ and $(g, \widetilde{X})$ are polarized pairs. Let τ be the central element of $(g, \widetilde{X})$. If $(h, \widetilde{X})$ is anticoherent to $(g, \widetilde{X})$ then $h = g^{-1} \cdot \tau$.*

Proof. By assumption, $\exists b \in B_n$ such that $g = h_{\tilde{b}}$ and $X = X_b$, reversing polarization. Thus $X_{bX^{-1}} = X$, preserving the polarization. Thus $(h_{\tilde{b}\widetilde{X}^{-1}}, \widetilde{X})$ is coherent with $(h_{\tilde{b}\widetilde{X}^{-1}}, \widetilde{X}_{\tilde{b}\widetilde{X}^{-1}})$. Clearly, $(h, \widetilde{X})$ is coherent with $(h_{\tilde{b}\widetilde{X}^{-1}}, \widetilde{X}_{\tilde{b}\widetilde{X}^{-1}})$. From uniqueness, $h = h_{\tilde{b}\widetilde{X}^{-1}} = g_{\widetilde{X}^{-1}}$. Since τ is the central element of $(g, \widetilde{X})$, $\tau = g \cdot g_{\widetilde{X}^{-1}}$. Thus, $g_{X^{-1}} = g^{-1}\tau$. So $h = g^{-1}\tau$. □

From uniqueness we get simultaneous conjugation:

Corollary 3.5. *If $(a_i, \widetilde{X})$ is coherent with $(g_i, \widetilde{Y})$, $i = 1, 2$, then there exist $b \in \widetilde{B}$ such that $(a_i)_b = g_i$, $i = 1, 2$.*

Proof. Let b be the element of $\widetilde{B}_n$ such that $(a_1)_b = g_1$, $(\widetilde{X})_b = \widetilde{Y}$. Now, $((a_2)_b, (X)_b)$ is coherent with $(a_2, \widetilde{X})$. Since $(\widetilde{X})_b = \widetilde{Y}$, $((a_2)_b, \widetilde{Y})$ is coherent with $(a_2, \widetilde{X})$. The pair $(g_2, \widetilde{Y})$ is also coherent with $(a_2, \widetilde{X})$. From uniqueness, $(a_2)_b = g_2$. □

4. $\widetilde{B}_n$-action of half-twists

In this section we study the action of half-twists on prime elements. We compute it for the case when the half-twist is adjacent to the supporting half-twist (Proposition 4.1) and for the case in which it is transversal (Lemma 4.2). The action of a disjoint half-twist is part of the defining axioms of prime elements (Axiom (3)).

Proposition 4.1. *Let G be a $\widetilde{B}_n$-group. Let $(h, \widetilde{X})$ be a polarized pair of G with corresponding central element τ. Let T, Y be two orderly adjacent polarized half-twists in B_n. Denote by $\widetilde{Y}'$ the polarized half-twist obtained from $\widetilde{Y}$ by changing polarization (that is, $\widetilde{T}$, $\widetilde{Y}'$ are not orderly adjacent). Denote by*

$$L(T) = L_{(h,\widetilde{X})}(\widetilde{T}), \qquad L(Y) = L_{(h,\widetilde{X})}(\widetilde{Y}), \qquad L(Y') = L_{(h,\widetilde{X})}(\widetilde{Y}').$$

Then

(1) $L(T)_{\widetilde{T}^{-1}} = L(T)^{-1}\tau$;

(2) $L(T)_{\widetilde{Y}^{-1}} = L(T)L(Y)$;

(3) $L(T)_{(\widetilde{Y}')^{-1}} = L(Y')^{-1}L(T)$.

Proof. (1) By Lemma 2.1(1).

(2) Let $b \in B_n$ be such that $L(T) = h_{\tilde{b}}$ and $T = X_b$, preserving the polarization. Let $Y_1 = Y_{b^{-1}}$. Then X, Y_1 are adjacent half-twists ($X = T_{b^{-1}}$, $Y_1 = Y_{b^{-1}}$), and so

$$h_{\widetilde{Y}_1^{-1}\widetilde{X}^{-1}} = h^{-1}h_{\widetilde{Y}_1^{-1}}.$$

Applying $\tilde{b}$ to that equation, we get

$$\big(L(T)\big)_{\widetilde{Y}^{-1}\widetilde{T}^{-1}} = L(T)^{-1}L(T)_{\widetilde{Y}^{-1}},$$

or

$$L(T)_{\widetilde{Y}^{-1}} = L(T) \cdot L(T)_{\widetilde{Y}^{-1}\widetilde{T}^{-1}}.$$

Let $b_1 = bY^{-1}T^{-1}$. Then $X_{b_1} = X_{bY^{-1}T^{-1}} = T_{Y^{-1}T^{-1}} = Y$ ($T_{Y^{-1}} = Y_T$ since Y, T are adjacent). Since T, Y are orderly adjacent and $X_b = T$, preserving the polarization, one can easily check that actually $X_{b_1} = Y$, preserving the polarization. Due to *uniqueness* of $L_{(h,\widetilde{X})}(\widetilde{Y}) = L(Y)$, we get $L(Y) = h_{\tilde{b}_1} = h_{\tilde{b}\widetilde{Y}^{-1}\widetilde{T}^{-1}} = L(T)_{\widetilde{Y}^{-1}\widetilde{T}^{-1}}$. Together with the previous equation we get: $L(T)_{\widetilde{Y}^{-1}} = L(T)L(Y)$, which is (2).

(3) From (2) we get

$$L(T)_{\widetilde{Y}'^{-1}} = L(T)_{\widetilde{Y}^{-1}} = L(T)L(Y).$$

By Lemma 2.1 $[L(T), L(T)_{\widetilde{Y}^{-1}}] = \tau$. Using (2) we substitute $L(T)L(Y)$ instead of $L(T)_{\widetilde{Y}^{-1}}$ and get

$$\tau = \big[L(T), L(T)L(Y)\big] = \big[L(T), L(Y)\big]$$

and

$$L(T)L(Y) = \tau L(Y)L(T).$$

Thus $L(T)_{\widetilde{Y}'^{-1}} = \tau L(Y)L(T)$. Using $Y_{Y^{-1}} = Y'$ (preserving the polarization) and uniqueness, we can write $L(Y') = L(Y)_{\widetilde{Y}^{-1}}$. By (1), $L(Y)_{\widetilde{Y}^{-1}} = L(Y)^{-1}\tau$, and thus $L(Y') = L(Y)^{-1}\tau$. As a consequence, we get $L(T)_{\widetilde{Y}'^{-1}} = L(Y')^{-1}L(T)$, which is (3). □

Lemma 4.2. *Let G be a $\widetilde{B}_n$-group. Let $h \in G$ be a prime element with supporting half-twist X. Let Z be a half-twist in B_n transversal to X. Then $h_{\widetilde{Z}} = h$.*

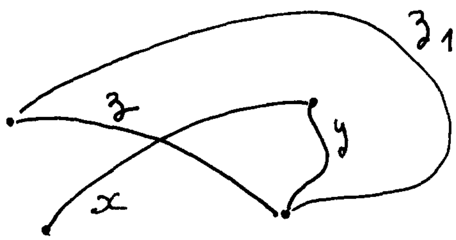


Fig. 8.

Proof. Let x, z be two transversally intersecting simple paths corresponding to X, Z (see Fig. 8).

There exists a simple path y such that the corresponding half-twist Y is adjacent to X and Z, and $Z_1 = Z_{Y^{-2}}$ is disjoint from X. Let z_1 be the path corresponding to Z_1 (see Fig. 8). We have

$$h_{\widetilde{Z}} = h_{\widetilde{Y}^{-2}\widetilde{Z}_1\widetilde{Y}^2} \overset{\text{Lemma 2.1}}{=} (h\tau)_{\widetilde{Z}_1\widetilde{Y}^2} = (h\tau)_{\widetilde{Y}^2} = h\tau\cdot\tau = h. \qquad \square$$

5. Commutativity properties

In this section we compute the commutator of prime elements depending on the relative location of their supporting half-twists. We also prove a commutativity result for $\widetilde{P}_n$ (the image of P_n, the pure braid group, in $\widetilde{B}_n$) to be used in studying the structure of $\widetilde{P}_n$ as a $\widetilde{B}_n$-group in Section 6 (see Lemma 6.1).

Proposition 5.1. *Let G be a $\widetilde{B}_n$-group. Let $(g_1, \widetilde{Y}_1)$, $(g_2, \widetilde{Y}_2)$ be two polarized pairs of G. Assume that they are coherent or anticoherent. Let τ be the corresponding central element of $(g_1, \widetilde{Y}_1)$ ($\tau = g_1(g_1)_{\widetilde{Y}_1^{-1}}$). Then*

(1) *if $\widetilde{Y}_1$, $\widetilde{Y}_2$ are adjacent, then $[g_1, g_2] = \tau$;*
(2) *if $\widetilde{Y}_1$, $\widetilde{Y}_2$ are disjoint or transversal, then $[g_1, g_2] = 1$.*

Proof. We use in the proof a few identities concerning commutators in a group:

$$[a, bc] = [a, b][a, c]_{b^{-1}}, \qquad [a, ba] = \left[a, b^{-1}\right]_{b^{-1}}^{-1} = [a, b],$$
$$\left[a, a^{-1}b\right] = [a, b]_a = \left[b, a^{-1}\right].$$

(1) Assume first that $(g_1, \widetilde{Y}_1), (g_2, \widetilde{Y}_2)$ are coherent. Take $b \in \widetilde{B}_n$ with $g_2 = (g_1)_b$, $\widetilde{Y}_2 = (\widetilde{Y}_1)_b$ (preserving the polarization). Let $b_1 = \widetilde{Y}_2^{-1}\widetilde{Y}_1^{-1}$. By Claim 1.1(b) $(\widetilde{Y}_1)_{b_1} = \widetilde{Y}_2$. Assume that b_1 preserves the polarization of $\widetilde{Y}_1$, $\widetilde{Y}_2$. We have $((g_1)_{b_1}, \widetilde{Y}_2)$ and $(g_2, \widetilde{Y}_2)$ coherent with $(g_1, \widetilde{Y}_1)$. By Theorem 3.3 (the *uniqueness* part) we get $(g_1)_{b_1} = g_2$. Thus we have

$$g_2 = (g_1)_{b_1} = (g_1)_{\widetilde{Y}_2^{-1}\widetilde{Y}_1^{-1}} \overset{\text{Axiom (2)}}{=} g_1^{-1}(g_1)_{\widetilde{Y}_2^{-1}},$$

and

$$[g_1,g_2]=[g_1,g_1^{-1}(g_1)_{\widetilde{Y}_2^{-1}}]=[g_1,(g_1)_{\widetilde{Y}_2^{-1}}]_{g_1}\overset{\text{Lemma 2.1(3)}}{=}\tau_{g_1}=\tau.$$

If b_1 does not preserve the polarization of $\widetilde{Y}_1,\widetilde{Y}_2$, then $b_2=b_1\widetilde{Y}_2$ does preserve it. As above, we get

$$g_2=(g_1)_{b_2}=(g_1)_{\widetilde{Y}_2^{-1}\widetilde{Y}_1^{-1}\widetilde{Y}_2}=(g_1)_{\widetilde{Y}_1\widetilde{Y}_2^{-1}\widetilde{Y}_1^{-1}}=(g_1^{-1}\tau)_{\widetilde{Y}_2^{-1}\widetilde{Y}_1^{-1}}=\tau(g_1^{-1})_{\widetilde{Y}_2^{-1}}g_1,$$

and then

$$\begin{aligned}[g_1,g_2]&=[g_1,\tau(g_1^{-1})_{\widetilde{Y}_2^{-1}}g_1]=[g_1,(g_1^{-1})_{\widetilde{Y}_2^{-1}}]\\&=[g_1,(g_1)_{\widetilde{Y}_2^{-1}}]^{-1}_{(g_1)_{\widetilde{Y}_2^{-1}}}\overset{\text{Lemma 2.1(3)}}{=}\tau.\end{aligned}$$

Assume now that $(g_1,\widetilde{Y}_1)$, $(g_2,\widetilde{Y}_2)$ are anticoherent. Denote by $\widetilde{Y}_2'$ the half-twist obtained from $\widetilde{Y}_2$ by changing polarization. One can then check that $(g_1,\widetilde{Y}_1)$, $((g_2)_{\widetilde{Y}_2^{-1}},\widetilde{Y}_2')$ are coherent. By the above $\tau=[g_1,(g_2)_{\widetilde{Y}_2^{-1}}]$. By Corollary 3.1, τ is also the central element of $(g_2.\widetilde{Y}_2)$. Thus $\tau=g_2(g_2)_{\widetilde{Y}_2^{-1}}$, which implies $(g_2)_{\widetilde{Y}_2^{-1}}=g_2^{-1}\tau$. Thus

$$\tau=[g_1,(g_2)_{\widetilde{Y}_2^{-1}}]=[g_1,g_2^{-1}\tau]=[g_1,g_2^{-1}]=[g_1,g_2]^{-1}_{g_2}.$$

Thus, $[g_1,g_2]=\tau_{g_2}^{-1}=\tau$.

(2) We can assume that $(g_1,\widetilde{Y}_1)$, $(g_2,\widetilde{Y}_2)$ are coherent. (Otherwise, we replace $\widetilde{Y}_2$ by $\widetilde{Y}_2'$ and g_2 by $(g_2)_{\widetilde{Y}_2^{-1}}$ and use $[g_1,(g_2)_{\widetilde{Y}_2^{-1}}]=[g_1,g_2]^{-1}_{g_2}$ as in (1).)

Consider first the case where $\widetilde{Y}_1$, $\widetilde{Y}_2$ are disjoint. We can choose a standard base of $\widetilde{B}_n$, say $(\widetilde{X}_1,\widetilde{X}_2,\ldots,\widetilde{X}_{n-1})$ such that $\widetilde{X}_1=\widetilde{Y}_1$, $\widetilde{X}_3=\widetilde{Y}_2$ and the given polarizations of $\widetilde{Y}_1$, $\widetilde{Y}_2$ coincide with "consecutive" polarizations of $\widetilde{X}_1$, $\widetilde{X}_3$ ("end" of $\widetilde{X}_1$ = "origin" of $\widetilde{X}_2$, "end" of $\widetilde{X}_2$ = "origin" of $\widetilde{X}_3$). Let $b_1=\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}$. By the proof of Claim 1.1, $\widetilde{Y}_2=(\widetilde{Y}_1)_{b_1}$, preserving the polarization. Then $((g_1)_{b_1},Y_2)=((g_1)_{b_1},(Y_1)_{b_1})$ is coherent with (g_1,Y_1). From Theorem 3.3 (*uniqueness*) it follows that

$$\begin{aligned}g_2=(g_1)_{b_1}&=(g_1)_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}}=(g_1^{-1}(g_1)_{\widetilde{X}_2^{-1}})_{\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}}\\&=(g_1^{-1})_{\widetilde{X}_2^{-1}}(g_1)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}}=(g_1^{-1})_{\widetilde{X}_2^{-1}}(g_1)_{\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}\\&=(g_1^{-1})_{\widetilde{X}_2^{-1}}(g_1)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}.\end{aligned}$$

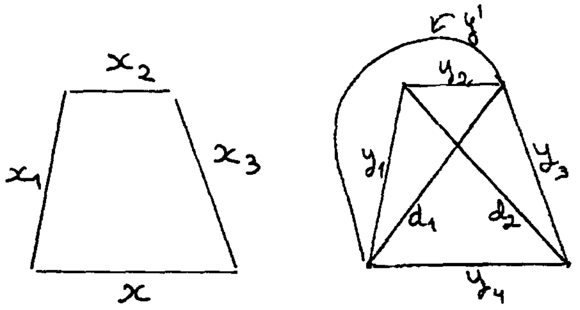
We can write

$$[g_1,g_2]=[g_1,(g_1^{-1})_{\widetilde{X}_2^{-1}}(g_1)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}]=[g_1,(g_1^{-1})_{\widetilde{X}_2^{-1}}]\cdot[g_1,(g_1)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}]_{(g_1)_{\widetilde{X}_2^{-1}}}.$$

By Lemma 2.1(3) $[g_1,(g_1)_{\widetilde{X}_2^{-1}}]=\tau$, which implies

$$[g_1,(g_1)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}]=[g_1,(g_1)_{\widetilde{X}_2^{-1}}]_{\widetilde{X}_3^{-1}}=\tau_{\widetilde{X}_3^{-1}}=\tau.$$

We also have $[g_1,(g_1^{-1})_{X_2^{-1}}]=[g_1,(g_1)_{X_2^{-1}}]^{-1}_{(g_1)_{X_2^{-1}}}=\tau^{-1}_{(g_1)_{X_2^{-1}}}=\tau^{-1}$. Thus $[g_1,g_2]=\tau\cdot\tau^{-1}=1$.



(a) (b)

Fig. 9.

Assume now that $\widetilde{Y}_1, \widetilde{Y}_2$ are transversal. As in the proof of Lemma 4.2, we can find a half-twist $\widetilde{T} \in \widetilde{B}_n$ such that $\widetilde{T}$ is adjacent to $\widetilde{Y}_1, \widetilde{Y}_2$ and $\widetilde{Y}_2' = (\widetilde{Y}_2)_{\widetilde{T}^{-2}}$ is disjoint from $\widetilde{Y}_1$. Let $b \in \widetilde{B}_n$ be such that $\widetilde{Y}_2 = (\widetilde{Y}_1)_b$, $g_2 = (g_1)_b$. Let $g_2' = (g_1)_{bT^2} = (g_2)_{\widetilde{T}^{-2}}$. Then $(g_2', \widetilde{Y}_2')$ is coherent with $(g_1, \widetilde{Y}_1)$. Since $\widetilde{Y}_2', \widetilde{Y}_1$ are disjoint, we get from the above $[g_1, g_2'] = 1$. By Lemma 2.1 $g_2' = (g_2)_{\widetilde{T}^{-2}} = g_2\tau$, or $g_2 = g_2'\tau$. Therefore, $[g_1, g_2] = [g_1, g_2'\tau] = [g_1, g_2'] = 1$. □

Note. Recall that there exists a natural homomorphism $\psi_n : B_n \to S_n$ where $\psi_n(X_i)$ is the transposition $(i, i+1)$ for a frame $\{X_i\}_{i=1}^{n-1}$ of $\widetilde{B}_n$. Its kernel $P_n = \ker\psi_n$ is called the pure braid group. Recall from [2,3] that P_n is generated by $\{Z_{ij}^2\}$, where $Z_{ij} = (X_i)_{X_{i+1}\cdots X_{j-1}}$.

Definition ($\widetilde{P}_n$). $\widetilde{P}_n = \ker(\widetilde{B}_n \overset{\widetilde{\psi}_n}{\to} S_n)$, where $\widetilde{\psi}_n$ is induced naturally from ψ_n $(n \geqslant 4)$.

Proposition 5.2. *Assume $n \geqslant 4$. Let $\widetilde{X}_1$, $\widetilde{X}_2$ be two adjacent half-twists in $\widetilde{B}_n$. Let $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$. Then the commutant $\widetilde{P}_n'$ of $\widetilde{P}_n$ is generated by c where $c_b = c$ $\forall b \in \widetilde{B}_n$, and $c^2 = 1$. Moreover, if $(\widetilde{Z}_1, \widetilde{Z}_2)$ is another pair of adjacent half-twists, then*

$$[\widetilde{Z}_1^2, \widetilde{Z}_2^2] = [\widetilde{Z}_1^2, \widetilde{Z}_2^{-2}] = [\widetilde{Z}_1^{-2}, \widetilde{Z}_2^{-2}] = c.$$

In particular, if $(\widetilde{Y}_1, \widetilde{Y}_2)$ and $(\widetilde{Z}_1, \widetilde{Z}_2)$ are two pairs of adjacent half-twists then $[\widetilde{Z}_1^2, \widetilde{Z}_2^2] = [\widetilde{Y}_1^2, \widetilde{Y}_2^2]$.

Proof. Let $B_n = B_n[D, K]$. Complete $\widetilde{X}_1$ and $\widetilde{X}_2$ to $\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}$, a standard base of $\widetilde{B}_n$, i.e., $X_i = H(x_i)$ and $x_1, \ldots, x_{n-1}$ are simple consecutive paths in D. Let $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$. Let $X = (X_1)_{\widetilde{X}_2\widetilde{X}_3}$. By Claim 1.0(iii) repeated twice, $H = H(x)$ where x is as in Fig. 9(a). We have a quadrangle formed by x_1, x_2, x_3, x (see Fig. 9(a)).

Evidently, $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}$ form a good quadrangle in $\widetilde{B}_n$. Thus by Lemma 1.2

$$\widetilde{X}_1^2\widetilde{X}_3^2 = \widetilde{X}_2^2\widetilde{X}^2. \tag{5.1}$$

Denote $y_1 = \widetilde{X}_1^2$, $y_2 = \widetilde{X}_2^2$, $y_3 = \widetilde{X}_3^2$, $y_4 = \widetilde{X}^2$ (the squares of the edges), $d_1 = \widetilde{X}_1\widetilde{X}_2^2\widetilde{X}_1^{-1}$, $d_2 = \widetilde{X}_2\widetilde{X}_3^2\widetilde{X}_2^{-1}$ (the squares of the diagonals), $y' = \widetilde{X}_2\widetilde{X}_1^2\widetilde{X}_2^{-1}$ (the square of the outer diagonal). See Fig. 9(b) where we denote the paths corresponding to the half-twists whose squares we considered here. Clearly,

$$y_1y_3 = y_3y_1, \qquad d_1 = (y')_{y_1^{-1}}, \qquad (y_3)_{x_2} = (d_2)_{x_3^{-2}} = (d_2)_{y_3^{-1}},$$

$$y_1y_2d_1 = y_2y_1y' = \Delta_3^2 \quad \text{(a central element of } P_3\text{)}.$$

We rewrite (5.1) to get

$$y_4 = y_1y_3y_2^{-1}. \tag{5.2}$$

Conjugating (5.2) by $\widetilde{X}_1$, we get

$$d_2 = y_1y_3y'^{-1}; \tag{5.3}$$

conjugating (5.2) by $\widetilde{X}_2$ we get $y_4 = d_1(d_2)_{y_3^{-1}} \cdot y_2^{-1}$. Since $d_1 = y_1y'y_1^{-1}$,

$$y_4 = y_1y'y_1^{-1}y_3d_2y_3^{-1}y_2^{-1} \overset{(5.3)}{=} y_1y'y_1^{-1}y_3y_1y_3y'^{-1}y_3^{-1}y_2^{-1}.$$

We compare the last expression with (5.2) to get

$$y'y_3^2y'^{-1}y_3^{-1} = y_3, \quad \text{or} \quad [y', y_3^2] = 1. \tag{5.4}$$

Since y', y_3 are squares of two adjacent half-twists in $\widetilde{B}_n$, and any two pairs of adjacent half-twists are conjugate, we conclude from (5.4) that:

$$\text{If } \widetilde{Z}_1, \widetilde{Z}_2 \text{ is a pair of adjacent half-twists in } \widetilde{B}_n,\ [\widetilde{Z}_1^2, \widetilde{Z}_2^4] = 1, \tag{5.5}$$

which also implies that $[\widetilde{Z}_1^2, \widetilde{Z}_2^2] = [\widetilde{Z}_1^{-2}, \widetilde{Z}_2^2] = [\widetilde{Z}_1^2, \widetilde{Z}_2^{-2}] = [\widetilde{Z}_1^{-2}, \widetilde{Z}_2^{-2}]$.

Conjugating (5.2) by $\widetilde{X}_3^{-1}$ we get

$$d_1 = y_1y_3(d_2)_{y_3^{-1}} = y_1y_3 \cdot y_3d_2^{-1}y_3^{-1} = y_1y_3^2 \cdot d_2^{-1}y_3^{-1} \overset{(5.5)}{=} y_1d_2^{-1}y_3.$$

We have by (5.5) that $1 = [y_1^2, y_2] = [y_1, y_2]_{y_1^{-1}} \cdot [y_1, y_2]$. Denoting $c = [y_1, y_2]$, we can write

$$c_{y_1^{-1}} = c^{-1}, \quad \text{or} \quad c_{y_1} = c^{-1}. \tag{5.6}$$

Denote by $\widetilde{P}_3$ the subgroup of $\widetilde{B}_n$ generated by y_1, y_2, d_1, and by $\alpha = \Delta_3^2 = y_1y_2d_1 = y_2y_1y'$ (a central element of $\widetilde{P}_3$), so that

$$y' = y_1^{-1}y_2^{-1}\alpha. \tag{5.7}$$

So,

$$\begin{aligned} c_{\widetilde{X}_1} &= [y_1, y_2]_{\widetilde{X}_1} = [y_1, y'] \overset{(5.7)}{=} [y_1, y_1^{-1}y_2^{-1}\alpha] \\ &= y_1 \cdot y_1^{-1}y_2^{-1}\alpha \cdot y_1^{-1} \cdot \alpha^{-1}y_2y_1 = y_2^{-1}y_1^{-1}y_2y_1 = [y_2^{-1}, y_1^{-1}] = [y_2, y_1] = c^{-1}. \end{aligned}$$

Thus we have $c_{\widetilde{X}_1^2} = (c^{-1})_{\widetilde{X}_1} = c$. By (5.6) $c_{\widetilde{X}_1^2} = c_{y_1} = c^{-1}$.

We compare the last two results to get $c = c^{-1}$ or

$$c^2 = 1 \quad \text{and} \quad c_{\widetilde{X}_1} = c. \tag{5.8}$$

Using a conjugation which sends $(\widetilde{X}_1, \widetilde{X}_2)$ to $(\widetilde{X}_2, \widetilde{X}_1)$, we obtain from (5.8)

$$c^{-1}_{\widetilde{X}_2} = c^{-1}, \quad \text{or} \quad c_{\widetilde{X}_2} = c. \tag{5.9}$$

(5.8) and (5.9) show that $\forall z \in \widetilde{B}_3$ (the subgroup of $\widetilde{B}_n$ generated by $\widetilde{X}_1$, $\widetilde{X}_2$) we have

$$c_z = c. \tag{5.10}$$

Consider now

$$\begin{aligned} c_{\widetilde{X}_3} &= [y_1, y_2]_{\widetilde{X}_3} = [y_1, d_2] \overset{(5.3)}{=} [y_1, y_1 y_3 y'^{-1}] \\ &= y_1 y_1 y_3 y'^{-1} y_1^{-1} y' y_3^{-1} y_1^{-1} = y_1 y_3 \cdot (y_1 y'^{-1} y_1^{-1} y') y_3^{-1} y_1^{-1} \\ &\overset{(5.10)}{=} y_1 y_3 c y_3^{-1} y_1^{-1} = c_{y_1^{-1} y_3^{-1}} = c_{y_1^{-3}} = c_{y_3} = c_{\widetilde{X}_3^2}, \end{aligned}$$

in short $c_{\widetilde{X}_3^2} = c_{\widetilde{X}_3}$. This implies $c_{\widetilde{X}_3} = c$.

Since $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$, we have $\forall \widetilde{X}_j$, $j \geqslant 4$, $c_{\widetilde{X}_j} = c$. Therefore, $\forall j$, $c_{\widetilde{X}_j} = c$, and thus $\forall b \in \widetilde{B}_n \; c_b = c$.

Let $(\widetilde{Y}_1, \widetilde{Y}_2)$ be another pair of adjacent half-twists. Since every 2 pairs of adjacent half-twists are conjugate in $\widetilde{B}_n$, $\exists b \in \widetilde{B}_n$ such that $[\widetilde{Y}_1^2, \widetilde{Y}_2^2] = [\widetilde{X}_1, \widetilde{X}_2^2]_b = c_b$. Since $c_b = c \; \forall b \in \widetilde{B}_n$, $[\widetilde{Y}_1^2, \widetilde{Y}_2^2] = c$. Since $c^2 = 1$, $c \in \text{Center}(\widetilde{B}_n)$, we also have $[\widetilde{Y}_1^2, \widetilde{Y}_2^{-2}] = [\widetilde{Y}_1^{-2}, \widetilde{Y}_2^{-2}] = c$. In particular, if $(\widetilde{Z}_1, \widetilde{Z}_2)$ is another pair of adjacent half-twists, $[\widetilde{Z}_1^2, \widetilde{Z}_2^2] = [\widetilde{Y}_1^2, \widetilde{Y}_2^2] = c$. Because any two disjoint and transversal half-twists of $\widetilde{B}_n$ commute, and $\widetilde{P}_n$ is generated by $\widetilde{Z}_{ij}^2 = (\widetilde{X}_i^2)_{\widetilde{X}_{i+1} \cdots \widetilde{X}_{j-1}}$, $1 \leqslant i < j \leqslant n$, we conclude that $\widetilde{P}_n'$ is generated by c. $\square$

Remark 5.3. Throughout the rest of the paper we will use c for the generator of $\widetilde{P}_n'$. We also recall from the above proposition that $c = [\widetilde{X}_1^2, X_2^2]$ for any two consecutive half-twists in B_n, $c \in \text{Center}\, B_n$ and $c^2 = 1$.

6. On the structure of $\widetilde{B}_n$ and $\widetilde{P}_n$

In Proposition 6.4 we establish a $\widetilde{B}_n$-isomorphism between $\widetilde{P}_n$ and the $\widetilde{B}_n$-group, $G(n)$, whose definition and structure are given before the proposition (Claim 6.2). Proposition 6.4 will be used in the final section when we give the simplest criterion for a prime element. Corollary 6.5 establishes the fact that $\widetilde{B}_n$ is an extension of a solvable group by a symmetric group, which is a structure type theorem for $\widetilde{B}_n$.

If G is a group, we denote its abelianization by $\text{Ab}(G)$. Recall that $\text{Ab}(B_n) \simeq \mathbb{Z}$. (This follows from the fact that B_n is generated by the half-twists and every two half-twists are conjugate to each other (Claim 1.1(c)).

Definition ($P_{n,0}$). $P_{n,0} = \ker(P_n \to \text{Ab}\, B_n)$ ("degree zero" pure braids).

Definition ($\widetilde{P}_{n,0}$). $\widetilde{P}_{n,0}$ is the image of $P_{n,0}$ in $\widetilde{P}_n$.

Remark 6.0. Let $X_1, \ldots, X_{n-1}$ be a frame of B_n, $Z_{ij} = (X_i)_{X_{i+1}\cdots X_{j-1}}$. Since P_n is generated by $\{Z_{ij}^2\}_{i<j}$ (see [2]), $\widetilde{P}_{n,0}$ is generated by $\{\widetilde{X}_1^2 \widetilde{Z}_{ij}^{-2}\}_{i<j}$. If $\tilde{g}_1, \ldots, \tilde{g}_m$ is another set of generators for $\widetilde{P}_{n,0}$, then $\{\widetilde{X}_1^2, \tilde{g}_1, \ldots, \tilde{g}_m\}$ generates $\widetilde{P}_n$.

Lemma 6.1. *Let X_1, X_2 be two consecutive half-twists in B_n. Let $u = (\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}} \widetilde{X}_2^{-2}$. Then $u \in \widetilde{P}_{n,0}$, u is a prime element in $\widetilde{P}_n$ (considered as a $\widetilde{B}_n$-group), $\widetilde{X}_1$ is its supporting half-twist and $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$ is its corresponding central element.*

Proof. Clearly, $u \in \widetilde{P}_{n,0}$. We often use here the fact that $(X_1)_{X_2^{-1}} = (X_2)_{X_1}$ as well as the fact that $[\widetilde{X}_1^{\pm 2}, \widetilde{X}_2^{\pm 2}] = c$, i.e., $\widetilde{X}_1^{-2}\widetilde{X}_2^2 = c\widetilde{X}_2^2\widetilde{X}_1^{-2}$ and $(\widetilde{X}_1^2)_{X_2^{-1}} = (\widetilde{X}_1^2)_{X_2} c$ for $c \in \mathrm{Center}(\widetilde{B}_n)$, $c^2 = 1$. In particular, $u = (\widetilde{X}_1)^2_{\widetilde{X}_2^{-1}} \widetilde{X}_2^{-2} = (\widetilde{X}_2)^2_{\widetilde{X}_1} \widetilde{X}_2^{-2}$. Complete X_1, X_2 to a frame of B_n: $X_1, \ldots, X_{n-1}$. ($\langle X_i, X_{i+1}\rangle = 1$ and $[X_i, X_j] = 1$, $|i-j| > 2$). We shall use Lemma 2.4, and so we must check conditions (1), (2a), (2b), (3) of Lemma 2.4.

(1) We have $u_{\widetilde{X}_1^{-1}} = (\widetilde{X}_2^2)_{\widetilde{X}_1\widetilde{X}_1^{-1}} \cdot (\widetilde{X}_2^{-2})_{\widetilde{X}_1^{-1}} = \widetilde{X}_2^2 \cdot (\widetilde{X}_1^{-2})_{\widetilde{X}_2}$. Let $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$. By Proposition 5.2, $(\widetilde{X}_1^{-2})_{\widetilde{X}_2} = (\widetilde{X}_1^{-2})_{\widetilde{X}_2^{-1}} c$. Thus $u_{\widetilde{X}_1^{-1}} = \widetilde{X}_2^2 \cdot (\widetilde{X}_1^{-2})_{\widetilde{X}_2^{-1}} c = u^{-1}c$. By Proposition 5.2, we also know that $c_b = c$ $\forall b \in \widetilde{B}_n$, $c \in \mathrm{Center}\,\widetilde{B}_n$ and $c^2 = 1$.

(2a) Since $[\widetilde{X}_1^2, \widetilde{X}_2^2] = c$, $u_{\widetilde{X}_2^{-1}} = (\widetilde{X}_1^2)_{\widetilde{X}_2^{-2}} \cdot \widetilde{X}_2^{-2} = c\widetilde{X}_1^2\widetilde{X}_2^{-2} = \widetilde{X}_2^{-2} \cdot \widetilde{X}_1^2$, and $u_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = (\widetilde{X}_2^{-2})_{\widetilde{X}_1^{-1}} \cdot \widetilde{X}_1^2 = (\widetilde{X}_1^{-2})_{\widetilde{X}_2} \cdot \widetilde{X}_1^2 = c(\widetilde{X}_1^{-2})_{\widetilde{X}_2^{-1}} \cdot \widetilde{X}_1^2$.

On the other hand, $u^{-1}u_{\widetilde{X}_2^{-1}} = \widetilde{X}_2^2 \cdot (\widetilde{X}_1^{-2})_{\widetilde{X}_2^{-1}} \cdot \widetilde{X}_2^{-2} \cdot \widetilde{X}_1^2 = c(\widetilde{X}_1^{-2})_{\widetilde{X}_2^{-1}} \cdot \widetilde{X}_1^2$. We get

$$u_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = u^{-1}u_{\widetilde{X}_2^{-1}}.$$

(2b) Using (1), we can write

$$u_{\widetilde{X}_1\widetilde{X}_2^{-1}} = \left(u^{-1}c\right)_{\widetilde{X}_2^{-1}} = u^{-1}_{\widetilde{X}_2^{-1}} \cdot c.$$

Conjugating by $\widetilde{X}_1^{-1}$ and using (2a) we get,

$$u_{\widetilde{X}_1\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = \left(u^{-1}\right)_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} \cdot c = u^{-1}_{\widetilde{X}_2^{-1}} u \cdot c.$$

Using (1) again we write

$$u^{-1}_{\widetilde{X}_1} u_{\widetilde{X}_1\widetilde{X}_2^{-1}} = cu \cdot u^{-1}_{\widetilde{X}_2^{-1}} c = u^{-1}_{\widetilde{X}_2^{-1}} u \cdot c.$$

(We use $u_{X_2^{-1}} = X_2^{-2} \cdot X_1^2$ and $[u, u_{\widetilde{X}_2^{-1}}] = [(\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}} \cdot \widetilde{X}_2^{-2}, \widetilde{X}_2^{-2}\widetilde{X}_1^2] = c \cdot c \cdot c = c$.) Thus,

$$u_{\widetilde{X}_1\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = u^{-1}_{\widetilde{X}_1} u_{\widetilde{X}_1\widetilde{X}_2^{-1}}.$$

(3) Clearly, $\forall j \geqslant 4$, $u_{\widetilde{X}_j} = u$. We need to check that $u_{\widetilde{X}_3} = u$. Consider $u_{\widetilde{X}_3^{-1}} = (\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} (\widetilde{X}_2^{-2})_{\widetilde{X}_3^{-1}}$.

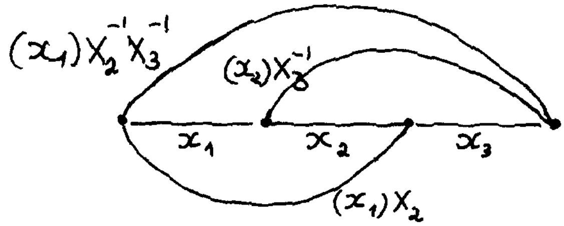


Fig. 10.

Since u can also be written as

$$u = (\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}} \cdot \widetilde{X}_2^{-2} = \widetilde{X}_2^{-2} \cdot (\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}} \cdot c = \widetilde{X}_2^{-2} \cdot (\widetilde{X}_1^2)_{\widetilde{X}_2},$$

we have:

$$u_{\widetilde{X}_3^{-1}} = u \Leftrightarrow (\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} \cdot (\widetilde{X}_2^{-2})_{\widetilde{X}_3^{-1}} = \widetilde{X}_2^{-2} \cdot (\widetilde{X}_1^2)_{\widetilde{X}_2}$$
$$\Leftrightarrow \widetilde{X}_2^2 \cdot (\widetilde{X}_1^2)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = (\widetilde{X}_1^2)_{\widetilde{X}_2} \cdot (\widetilde{X}_2^2)_{\widetilde{X}_3^{-1}}.$$

The last equality is valid since $\{(\widetilde{X}_1)_{\widetilde{X}_2}, \widetilde{X}_2, (\widetilde{X}_2)_{\widetilde{X}_3^{-1}}, (\widetilde{X}_1)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}\}$ form a good quadrangle. See Fig. 10 for the curves representing the quadrangle. (Recall that $(X_1)_{X_2} = H((x_1)X_2)$.) □

Construction of $\boldsymbol{G}(n)$

For $n \geqslant 3$ we define the group $\boldsymbol{G}(n)$ as follows:
Generators: $s_1,\ u_1,\ u_2,\ \ldots,\ u_{n-1}$.
Relations:

$$[s_1, u_i] = 1 \quad \forall i = 1, 3, \ldots, n-1;$$
$$[u_i, u_j] = 1 \quad \text{when } |i-j| \geqslant 2;$$
$$[s_1, u_2] = [u_i, u_{i+1}] \quad \forall i = 1, 2, \ldots, n-2;$$
$$[u_1, u_2] = [u_1, u_2]_{s_1} = [u_1, u_2]_{u_i} \quad \forall i = 1, 2, \ldots, n-1;$$
$$[u_1, u_2]^2 = 1.$$

We denote: $[u_1, u_2]$ by ν. Clearly, $\nu \in \mathrm{Center}(\boldsymbol{G}(n))$, $\boldsymbol{G}(n) = \{1, \nu\}$ and $\nu^2 = 1$.

Equivalent construction of $\boldsymbol{G}(n)$

Consider a free abelian group $A(n)$ with generators $S_1,\ V_1,\ \ldots,\ V_{n-1}$ and a skew-symmetric $\mathbb{Z}/2$-valued bilinear form $Q(x, y)$ on $A(n)$ defined by:

$$Q(S_1, V_i) = 0 \quad \forall i = 1, 3, \ldots, n-1;$$
$$Q(V_i, V_j) = 0 \quad \text{when } |i-j| \geqslant 2,$$
$$Q(S_1, V_2) = Q(V_i, V_{i+1}) = 1 \quad \forall i = 1, 2, \ldots, n-2.$$

One can check that there exists a unique central extension G of $A(n)$ by $\mathbb{Z}/2$ with $\mathrm{Ab}(G) \simeq A(n)$ $(G' \simeq \mathbb{Z}/2)$ and $[x,y] = Q(\overline{x},\overline{y})$, $\forall x,y \in G$ and $\overline{x}$, $\overline{y}$ the images of x, y in $A(n)$.

Claim 6.2. (1) *The above central extension is isomorphic to* $\boldsymbol{G}(n)$.

(2) $\mathrm{Ab}(\boldsymbol{G}(n))$ *is a free abelian group with* n *generators (i.e.,* $A(n)$*) and* $\boldsymbol{G}(n)' \simeq \mathbb{Z}/2$ *generated by* $\nu = [u_1, u_2]$.

(3) *The following formulas define a* $\widetilde{B}_n$*-action on* $\boldsymbol{G}(n)$ *for* $\{\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}\}$, *a frame of* $\widetilde{B}_n$.

$\widetilde{X}_1$-action	$\widetilde{X}_2$-action	$\widetilde{X}_k$-action, $k \geqslant 3$
$s_1 \to s_1$	$s_1 \to u_2 s_1$	$s_1 \to s_1$
$u_1 \to u_1^{-1}\nu$	$u_1 \to u_2 u_1$	$u_{k-1} \to u_k u_{k-1}$
$u_2 \to u_1 u_2$	$u_2 \to u_2^{-1}\nu$	$u_k \to u_k^{-1}\nu$
$u_j \to u_j \ \forall j \geqslant 3$	$u_3 \to u_2 u_3$;	$u_{k+1} \to u_k u_{k+1}$
	$u_j \to u_j \ \forall j \geqslant 4$	$u_j \to u_j \ \forall j \neq k-1, k, k+1$

(4) *Let* $b \in \widetilde{B}_n$, $y = (\widetilde{X}_1)_b$. *Then the* y^2*-action on* $\boldsymbol{G}(n)$ *coincides with the conjugation by* $(s_1)_b$.

(5) *Let*

$$s_{ij} = \begin{cases} s_1 & \textit{if } (i,j) = (1,2); \\ (s_1)_{\widetilde{X}_2 \cdots \widetilde{X}_{j-1}} \ \left(\overset{\text{Claim 6.2(3)}}{=} u_{j-1} \cdots u_2 s_1 \right) & \textit{if } i = 1,\ j \geqslant 2; \\ (s_1)_{\widetilde{X}_2 \cdots \widetilde{X}_{j-1} \widetilde{X}_1 \cdots \widetilde{X}_{i-1}} & \\ \quad = \begin{cases} \nu \cdot u_{j-1} \cdots u_1 \cdot u_{i-1} \cdots u_2 s_1 \\ \nu \cdot u_{j-1} \cdots u_1 s_1 \end{cases} & \begin{array}{l} \textit{if } i \geqslant 3,\ j > i; \\ \textit{if } i = 2,\ j > i. \end{array} \end{cases} \tag{6.1}$$

Then

$$[s_{ij}, s_{kl}] = \begin{cases} \nu & \textit{if } \{i,j\} \cap \{k,l\} = 1; \\ 1 & \textit{otherwise.} \end{cases}$$

(6) *Let* $\widetilde{F}_{n-1}$ *be the subgroup of* $\boldsymbol{G}(n)$ *generated by* $\{s_{n-1,n}, s_{n-2,n}, \ldots, s_{1n}\}$. *The generators* $\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}$ *of* $\widetilde{B}_{n-1}$ *act on* $\widetilde{F}_{n-1}$ *as follows:*

$$\begin{aligned} &(s_{jn})_{\widetilde{X}_k} = s_{jn} \quad j \neq k, k+1; \\ &(s_{k,n})_{\widetilde{X}_k} = s_{k+1,n}; \\ &(s_{k+1,n})_{\widetilde{X}_k} = s_{kn}\nu = s_{k+1,n} s_{kn} s_{k+1,n}^{-1} \end{aligned} \tag{6.2}$$

for $k = 1, \ldots, n-2$.

Their actions correspond to the standard Hurwitz moves on $(s_{n-1,n}, s_{n-1,n}, \ldots, s_{1n})$ *(see, for example,* [3, Chapter 4]*), and it defines a* $\widetilde{B}_{n-1}$*-action.*

(7) *There is a natural chain of embeddings* $\boldsymbol{G}(3) \subset \boldsymbol{G}(4) \subset \cdots \subset \boldsymbol{G}(n-1) \subset \boldsymbol{G}(n)$ *corresponding to the chain*

$$(s_1, u_1, u_2) \subset (s_1, u_1, u_2, u_3) \subset \cdots \subset (s_1, u_1, \ldots, u_{n-1}).$$

Proof. (1), (2), and (3) are easy to verify.

(4) Consider first the case $b = \mathrm{Id}$. From (3) we get for the $\widetilde{X}_1^2$-action:

$$s_1 \to s_1, \qquad u_1 \to u_1, \qquad u_2 \to u_2\nu, \qquad u_j \to u_j \quad \forall j \geqslant 3.$$

At the same time by the first construction

$$(s_1)_{s_1} = s_1, \qquad (u_1)_{s_1} = u_1, \qquad (u_2)_{s_1} = s_1^{-1}u_2 s_1 = u_2\nu,$$
$$(u_j)_{s_1} = u_j \quad \forall j \geqslant 3.$$

Thus $\widetilde{X}_1^2$-action and s_1-conjugation coincide. Consider now any $b \in \widetilde{B}_n$ and any $g \in \boldsymbol{G}(n)$. Let $h = g_{b^{-1}}$. We have

$$g_{(\widetilde{X}_1^2)_b} = g_{b^{-1}\widetilde{X}_1^2 b} = \big((h)_{\widetilde{X}_1^2}\big)_b = (h_{s_1})_b = (h_b)_{(s_1)_b} = g_{(s_1)_b}.$$

(5)–(7) are easy to verify. $\square$

Lemma 6.3. *Let* $n \geqslant 3$. *Let* $\{X_1, \ldots, X_{n-1}\}$ *be a frame of* B_n. *Let*

$$Z_{ij} = \begin{cases} X_1 & \text{if } (i,j) = (1,2); \\ (X_1)_{X_2 \cdots X_{j-1}} & \text{if } i = 1,\ j \geqslant 3; \\ (X_1)_{X_2 \cdots X_{j-1} X_1 \cdots X_{i-1}} & \text{if } i \geqslant 2,\ j > i. \end{cases}$$

Let $\widetilde{Z}_{ij}$ *be the image of* Z_{ij} *in* $\widetilde{B}_n$. *Consider* $\boldsymbol{G}(n)$ *as a* $\widetilde{B}_n$*-group as in Claim* 6.2.

Then there exists a unique $\widetilde{B}_n$*-surjection* $\Lambda_n : \widetilde{P}_n \to \boldsymbol{G}(n)$ *with* $\Lambda_n(\widetilde{X}_1^2) = s_1$ *and* $\Lambda_n(\widetilde{Z}_{ij}^2) = s_{ij}$ *for* $1 \leqslant i < j \leqslant n$. *(In particular,* $\Lambda_n(\widetilde{X}_1^2) = s_1$.*)*

Proof. Use induction on n.

By definition $\widetilde{P}_3$ is obtained from P_3 by adding the relations $[\widetilde{Z}_{12}^2, \widetilde{Z}_{23}^2] = [\widetilde{Z}_{12}^2, \widetilde{Z}_{13}^2] = [\widetilde{Z}_{23}^2, \widetilde{Z}_{13}^2]$ and $[\widetilde{Z}_{12}^2, \widetilde{Z}_{23}^2]^2 = 1$. By Proposition 5.2 the embedding $P_3 \subset P_4$ induces a homomorphism $\widetilde{P}_3 \xrightarrow{i_3} \widetilde{P}_4$. For $n = 3$, $\Lambda_3 : \widetilde{P}_3 \to \boldsymbol{G}(3)$ must be defined by $\lambda_3(\widetilde{Z}_{ij}^2) = s_{ij}$, $1 \leqslant i < j \leqslant 3$. One can check directly that Λ_3 is well defined, and that it is a $\widetilde{B}_3$-surjection. Uniqueness of such Λ_3 is evident.

Assume now that $n \geqslant 4$ and the desired $\Lambda_{n-1} : \widetilde{P}_{n-1} \to \boldsymbol{G}(n-1)$ exists.

We shall establish first a surjection $\widehat{\Lambda}_n : P_n \twoheadrightarrow \boldsymbol{G}(n)$. Considering

$$\{X_1, X_2\} \subset \{X_1, X_2, X_3\} \subset \cdots \subset \{X_1, \ldots, X_{n-1}\},$$

we get a chain of embeddings $B_3 \subset B_4 \subset \cdots \subset B_n$ and the corresponding chain $P_3 \subset P_4 \subset \cdots \subset P_n$. To the latter corresponds a chain of homomorphisms:

$$\widetilde{P}_3 \xrightarrow{i_3} \widetilde{P}_4 \xrightarrow{i_4} \cdots \to \widetilde{P}_{n-1} \xrightarrow{i_{n-1}} \widetilde{P}_n.$$

The Z_{ij}'s defined here are in fact the same as those defined in Remark 6.0 (by Claim 1.0(iii) and thus they generate P_n. For our proof it is better to use tlese definitions of Z_{ij}'s. It is also known that $P_n \simeq P_{n-1} \ltimes F_{n-1}$, where P_{n-1} is the subgroup of P_n generated by $\{Z_{ij}^2 \mid 1 \leqslant i < j \leqslant n-1\}$, F_{n-1} is the free subgroup of P_n generated by $\{Z_{in}^2 \mid 1 \leqslant i \leqslant n-1\}$, and the semidirect product $P_{n-1} \ltimes F_{n-1}$ is defined according to the P_{n-1}-action on F_{n-1} which comes from the B_{n-1}-action by conjugation (using $B_{n-1} \subset B_n \supset P_n$). The latter coincides with the standard B_{n-1}-action on F_{n-1} (the generators $X_{n-2}, \ldots, X_1$ of B_{n-1} correspond to standard *Hurwitz moves* on $(Z_{n-1,n}^2, Z_{n-2,n}^2, \ldots, Z_{1n}^2)$.

Using the canonical map $P_{n-1} \to \widetilde{P}_{n-1}$, we obtain from Λ_{n-1} a B_{n-1}-surjection $\widehat{\Lambda}_{n-1} : P_{n-1} \to \boldsymbol{G}(n-1)$. For the free subgroup F_{n-1} of P_n generated by $\{Z_{in}^2 \mid n-1 \leqslant i \leqslant 1\}$ define $\mu_{n-1} : F_{n-1} \to \boldsymbol{G}(n)$ by $\mu_{n-1}(Z_{in}^2) = s_{in}$. Considering P_n as $P_{n-1} \ltimes F_{n-1}$, we define $\widehat{\Lambda}_n : P_n \to \boldsymbol{G}(n)$, which on P_{n-1} coincmdes with

$$\widehat{\Lambda}_{n-1} : P_{n-1} \to \boldsymbol{G}(n-1) \subset \boldsymbol{G}(n)$$

(see Claim 6.2(7)) and on F_{n-1} coincides with $\mu_{n-1} : F_{n-1} \to \boldsymbol{G}(n)$. To show that such $\widehat{\Lambda}_n$ exists one has to check the following:

Claim 1. *The conjugation of $\mu_{n-1}(F_{n-1})$ by elements of $\widehat{\Lambda}_{n-1}(P_{n-1})$ ($\subset \boldsymbol{G}(n)$) coincides with the P_{n-1}-action defined by $P_{n-1} \subset B_{n-1} \subset B_n \to \widetilde{B}_n$ and the given $\widetilde{B}_n$-action on $\boldsymbol{G}(n)$. That is, $\forall f \in \mu_{n-1}(F_{n-1})$ and $\forall h$ of the form $\widehat{\Lambda}_{n-1}(\widetilde{Y})$ ($Y \in P_{n-1}$) we must have $h^{-1} f h = f_{\widetilde{Y}}$.*

Proof. Since $\forall b \in B_{n-1}$, $\widehat{\Lambda}_{n-1}((X_1^2)_b) = (s_1)_b$ we see from Claim 6.2 that $\forall f \in \mu_{n-1}(F_{n-1})$, $f_{\widehat{\Lambda}_{n-1}((X_1^2)_b)} = (f_{s_1})_b = f_{(\widetilde{X}_1^2)_b}$. Since P_{n-1} is generated by $\{Z_{ij}^2 \mid 1 \leqslant i < j \leqslant n-1\}$, i.e., by $\{(X_1^2)_b \mid b \in B_n\}$ we get (1). □

Claim 2. *The P_{n-1}-action on $\mu_{n-1}(F_{n-1})$ (defined by $P_{n-1} \subset B_{n-1} \subset B_n \to \widetilde{B}_n$ and the given $\widetilde{B}_n$-action on $\boldsymbol{G}, n)$) comes from the B_{n-1}-action on $\mu_{n-1}(F_{n-1})$ in which $X_{n-2}, \ldots, X_1$ correspond to the standard Hurwitz moves on $(s_{n-1,n}, s_{n-2,n}, \ldots, s_{1n})$.*

Proof. It follows immediately from Claim 6.2(6). □

Thus, Claims 1 and 2 are true and we can extend $\widehat{\Lambda}_{n-1}$, μ_{n-1} to a homomorphism

$$\widehat{\Lambda}_n : P_n \ (= P_{n-1} \ltimes F_{n-1}) \to \boldsymbol{G}(n).$$

For $1 \leqslant i < j \leqslant n-1$, $\widehat{\Lambda}_n(Z_{ij}^2) = \widehat{\Lambda}_{n-1}(Z_{ij}^2) = s_{ij}$, and for $1 \leqslant i \leqslant n-1$, $\widehat{\Lambda}_n(Z_{in}^2) = \mu_{n-1}(Z_{in}^2) = s_{in}$. Thus $\widehat{\Lambda}_n(Z_{ij}^2) = s_{ij}$ for $1 \leqslant i < j \leqslant n$.

Using induction, one can check directly that $\widehat{\Lambda}_n$ is a B_n-homomorphism (recall that by Claim 6.2 we have explicit formulas for s_{ij}'s).

Since $s_{1n} = u_{n-1} \cdot \cdots \cdot u_2 s_1$ (see (6.1)), $u_{n-1} = s_{1n}(u_{n-2} \cdots u_2 s_1)^{-1}$. Thus $\boldsymbol{G}(n)$ is generated by $\boldsymbol{G}(n-1)$ and s_{1n}. But $\boldsymbol{G}(n-1) = \widehat{\Lambda}_{n-1}(P_{n-1}) = \widehat{\Lambda}_n(P_{n-1})$ and $s_{1n} = \widehat{\Lambda}_n(Z_{1n}^2)$, and thus $\widehat{\Lambda}_n$ is a B_n-surjection.

Let $N = \ker(P_n \to \widetilde{P}_n)$ ($= \ker(B_n \to \widetilde{B}_n)$). Let $T = X_1^2 X_3^2 X_2^{-2} Z_{14}^{-2}$. By Claim 1.2(b), N is generated by $\{T_b \mid b \in B_n\}$. We have

$$\begin{aligned}
\widehat{\Lambda}_n(T) &= \widehat{\Lambda}_4(T) = s_1 \cdot s_{34} \cdot s_{23}^{-1} s_{14}^{-1} \\
&\overset{\text{Claim 6.2}}{=} s_1 \cdot \nu u_3 u_2 u_1 \cdot u_2 s_1 \cdot s_1^{-1} u_1^{-1} u_2^{-1} \nu \cdot s_1^{-1} u_2^{-1} u_3^{-1} \\
&= s_1 \nu u_3 u_2 \nu \cdot s_1^{-1} u_2^{-1} u_3^{-1} \\
&= \nu u_3 u_2 u_2^{-1} u_3^{-1} \\
&\qquad \text{(since } s_1 \text{ commutes with } u_3,\ [s_1, u_2] = \nu \in \operatorname{Center} \boldsymbol{G}(n) \text{ and } \nu^2 = 1) \\
&= \nu^2 \quad (\text{since } [u_3, u_2] = \nu) = \mathrm{Id}.
\end{aligned}$$

Since $\widehat{\Lambda}_n$ is a B_n-homomorphism, we get $\widehat{\Lambda}_n(T_b) = \mathrm{Id}\ \forall b \in B_n$, and thus $\widehat{\Lambda}_n(N) = \mathrm{Id}$. Hence $\widehat{\Lambda}_n$ defines canonically a $\widetilde{B}_n$-surjection $\Lambda_n : \widetilde{P}_n \to \boldsymbol{G}(n)$ with $\Lambda_n(\widetilde{X}_1^2) = s_1$.

Uniqueness of such Λ_n follows from the fact that $\widetilde{P}_n$ is generated by the $\widetilde{B}_n$-orbit of $\widetilde{X}_1^2$. □

Theorem 6.4. *There exists a unique $\widetilde{B}_n$-isomorphism $\Lambda_n : \widetilde{P}_n \to \boldsymbol{G}(n)$ with $\Lambda_n(c) = \nu$ (see Remark 5.3 and Claim 6.2), s.t. $\widetilde{P}_{n,0}$ is $\widetilde{B}_n$-isomorphic to the subgroup $G_0(n)$ of $\boldsymbol{G}(n)$, generated by $u_1, \ldots, u_{n-1}$. In particular:*

(1) $\mathrm{Ab}\,\widetilde{P}_n$ *is a free abelian group with n generators.*

(2) $\mathrm{Ab}\,\widetilde{P}_{n,0}$ *is a free abelian group generated by $n-1$ prime elements where c is the commutator of any 2 of them. $\widetilde{P}_n' = \widetilde{P}_{n,0}' \simeq \mathbb{Z}/2$, generated by c.*

(3) $\widetilde{P}_{n,0}$ *is a primitive $\widetilde{B}_n$-group generated by the $\widetilde{B}_n$-orbit of a prime element $u = \widetilde{X}_1^2 \widetilde{X}_2^{-2}$, where $\widetilde{X}_1, \widetilde{X}_2$ are consecutive half-twists in $\widetilde{B}_n$, $\widetilde{T} = \widetilde{X}_2^{-1} \widetilde{X}_1 \widetilde{X}_2$ is the supporting half-twist for u, and $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$ is its corresponding central element.*

Proof. We first prove (3), since we use it to prove that the surjection Λ_n from Lemma 6.3 is in fact an isomorphism. Complete $\widetilde{X}_1$, $\widetilde{X}_2$ to a frame $\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}$ of $\widetilde{B}_n$. We apply conjugation by $\widetilde{X}_2$ on Lemma 6.1 and conclude using Lemma 2.2 that u is prime, its supporting half-twist is $\widetilde{X}_2^{-1} \widetilde{X}_1 \widetilde{X}_2$ and its corresponding central element is c. By Remark 6.0, $\widetilde{P}_{n,0}$ is generated by $\{\widetilde{X}_1^2 \widetilde{Z}_{ij}^{-2} \mid l \leqslant i < j \leqslant n\}$. Since $\widetilde{X}_1^2 \cdot \widetilde{Z}_{ij}^{-2} = \widetilde{X}_1^2 \widetilde{Z}_{1i}^{-2} \cdot \widetilde{Z}_{1i}^2 \cdot \widetilde{Z}_{ij}^{-2}$ and both $\widetilde{X}_1^2 \widetilde{Z}_{1i}^{-2}$,$\widetilde{Z}_{1i}^2 \cdot \widetilde{Z}_{ij}^{-2}$ are conjugates of u (by Claim 1.1(f)), $\widetilde{P}_{n,0}$ is generated by the $\widetilde{B}_n$-orbit of u. Therefore, $\widetilde{P}_{n,0}$ is a primitive $\widetilde{B}_n$-group. Thus, we proved (3).

Polarize each X_i (and $\widetilde{X}_i$) according to the sequence $(X_1, \ldots, X_{n-1})$ (the "end" of X_i = the "origin" of X_{i+1}). By Theorem 3.3 $\forall i = 1, \ldots, n-1$, $\exists$ a unique prime element $\xi_i = L_{(u, \widetilde{X}_1)}(\widetilde{X}_i) \in \widetilde{P}_{n,0}$ such that $(\xi_i, \widetilde{X}_i)$ is coherent with $(u, \widetilde{X}_1)$. Clearly $\xi_1 = u$. By Lemma 3.1 the corresponding central element of ξ_i, $i = 1, \ldots, n$, is also c.

By Proposition 4.1(1) and (2) we have $\forall i = 1, \ldots, n-1$:

$$(\xi_i)_{\widetilde{X}_i^{-1}} = \xi_i^{-1} c, \qquad (\xi_i)_{\widetilde{X}_{i-1}^{-1}} = \xi_i \xi_{i-1}, \qquad (\xi_i)_{\widetilde{X}_{i+1}^{-1}} = \xi_i \xi_{i+1}. \tag{6.3}$$

It is also clear (Axiom (3)) that $\forall j \neq i, i-1, i+1$

$$(\xi_i)_{\widetilde{X}_j} = \xi_i. \tag{6.4}$$

We see from (6.3), (6.4) that the subgroup of $\widetilde{P}_{n,0}$ generated by $\{\xi_1, \dots, \xi_{n-1}\}$ is closed under the $\widetilde{B}_n$-action. Since $\widetilde{P}_{n,0}$ is generated by the $\widetilde{B}_n$-orbit of $u = \xi_1$, we conclude that $\widetilde{P}_{n,0}$ is generated by $\{\xi_1, \dots, \xi_{n-1}\}$. This implies (see Remark 6.0) that $\widetilde{P}_n$ is generated by $\{\widetilde{X}_1^2, \xi_1, \xi_2, \dots, \xi_{n-1}\}$.

Since X_1 and X_2 are consecutive, $\widetilde{X}_2 = (\widetilde{X}_1)_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}$ (Claim 1.0(ii)). By the *uniqueness* of Theorem 3.3,

$$\xi_2 \ (= L_{(\xi_1, \widetilde{X}_1)}(\widetilde{X}_2)) = (\xi_1)_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}$$

which equals $(\widetilde{X}_1^{-2})_{\widetilde{X}_2} \cdot \widetilde{X}_1^2$ ($\xi_1 = u$). Thus

$$[\widetilde{X}_1^2, \xi_2] = c. \tag{6.5}$$

By Lemma 5.1 we have

$$[\xi_i, \xi_j] = \begin{cases} c & \text{if } |i-j| = 1, \\ 1 & \text{if } |i-j| \geqslant 2. \end{cases} \tag{6.6}$$

Observe also that

$$(\widetilde{X}_1^2)_{\widetilde{X}_2} = \xi_2 \cdot \widetilde{X}_1^2. \tag{6.7}$$

Formulas (6.3)–(6.7) show that we can define a $\widetilde{B}_n$-homomorphism $M_n : \boldsymbol{G}(n) \to \widetilde{P}_n$ with $M_n(s_1) = \widetilde{X}_1^2$, $M_n(u_i) = \xi_i$, $i = 1, \dots, n-1$. (See Claim 6.2.)

Since $\widetilde{P}_n$ is generated by the $\widetilde{B}_n$-orbit of $\widetilde{X}_1^2$ and $\boldsymbol{G}(n)$ is generated by the $\widetilde{B}_n$-orbit of s_1, we conclude that Λ_n and M_n are inverses of each other. $\square$

The next corollary immediately follows from Theorem 6.4.

Corollary 6.5. (1) *We have the following sequence for* $\widetilde{B}_n$:

$$1 < (\widetilde{P}'_n =) \ \widetilde{P}'_{n,0} < \widetilde{P}_{n,0} < \widetilde{P}_n < \widetilde{B}_n$$

where:

$$\widetilde{B}_n / \widetilde{P}_n \simeq S_n, \qquad \widetilde{P}_n / \widetilde{P}_{n,0} \simeq \mathbb{Z},$$
$$\widetilde{P}_{n,0} / \widetilde{P}'_{n,0} \simeq \mathbb{Z}^{n-1}, \qquad \widetilde{P}'_{n,0} (= \widetilde{P}'_n) \simeq \mathbb{Z}_2.$$

(2) $\widetilde{B}_n$ *is an extension of a solvable group by a symmetric group. In particular, it is "almost solvable", i.e., it contains a solvable group with a finite index.*

7. Criterion for prime element

The criterion for an element of a $\widetilde{B}_n$-group to be prime, presented in this section (Proposition 7.1), is simpler to use than previous ones but its proof is much longer. The proof contains 8 lemmas. This criterion will be used in the application of this paper to study fundamental groups which turn out to be $\widetilde{B}_n$-groups.

Proposition 7.1. *Assume $n \geqslant 5$. Let G be a $\widetilde{B}_n$-group, $(\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_{n-1})$ be a frame of $\widetilde{B}_n$. Let S be an element of G with the following properties:*

(0) *G is generated by $\{S_b, b \in \widetilde{B}_n\}$;*

(1a) $S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = S^{-1} S_{\widetilde{X}_2^{-1}}$;

(1b) $S_{\widetilde{X}_1 \widetilde{X}_2^{-1} \widetilde{X}_1^{-1}} = S_{\widetilde{X}_1}^{-1} S_{\widetilde{X}_1 \widetilde{X}_2^{-1}}$;

(2) *For $\tau = S S_{\widetilde{X}_1^{-1}}$, $T = S_{\widetilde{X}_2^{-1}}$ we have:*

(2a) $\tau_{\widetilde{X}_1^2} = \tau$;

(2b) $\tau_T = \tau_{\widetilde{X}_1}^{-1}$;

(3) $S_{\widetilde{X}_j} = S \ \forall j \geqslant 3$;

(4) *$S_c = S$, where $c = [\widetilde{X}_1^2, \widetilde{X}_2^2]$.*

Then S is a prime element of G, $\widetilde{X}_1$ is the supporting half-twist of S and τ is the corresponding central element. In particular, $\tau^2 = 1$, $\tau \in \mathrm{Center}(G)$, $\tau_b = \tau \ \forall\, b \in \widetilde{B}_n$.

Proof. The proof includes several lemmas. From Theorem 5.2, $c \in \mathrm{Center}(\widetilde{B}_n)$, $c^2 = 1$. From Theorem 6.4 it follows that $\widetilde{P}'_n$ is generated by c. $\widetilde{P}'_n$ is a normal subgroup of $\widetilde{B}_n$. Write $\widetilde{\widetilde{B}}_n = \widetilde{B}_n / \widetilde{P}'_n$, $\widetilde{\widetilde{P}}_n = \widetilde{P}_n / \widetilde{P}'_n = \mathrm{Ab}\, \widetilde{P}_n$. Clearly, $\widetilde{\widetilde{P}}_n$ is a commutative group. We have $\widetilde{\widetilde{\psi}}_n : \widetilde{\widetilde{B}}_n \twoheadrightarrow S_n$. By abuse of notation we use ψ_n for $\widetilde{\widetilde{\psi}}_n$. Let $Y \in B_n$. By abuse of notation we denote the image of Y in $\widetilde{B}_n$ or in $\widetilde{\widetilde{B}}_n$ or in $\widetilde{\widetilde{P}}_n$ by the same symbol $\widetilde{Y}$. It is clear that $\widetilde{B}_n$ acts on $\widetilde{\widetilde{P}}_n$ (through conjugations) as the symmetric group $S_n = \widetilde{B}_n / \widetilde{P}_n$.

Since $S_c = S$ and $c \in \mathrm{Center}\, \widetilde{B}_n$, we have $\forall b \in \widetilde{B}_n$ $(S_b)_c = S_{bc} = (S_c)_b = S_b$. Since G is generated by $\{S_b,\ b \in \widetilde{B}_n\}$ we have $\forall g \in G$, $g_c = g$. In particular, we conclude that $\widetilde{B}_n$ acts on G as its quotient $\widetilde{\widetilde{B}}_n$; in other words, G is a $\widetilde{\widetilde{B}}_n$-group.

Let (D, K) be a model for B_n, $K = \{a_1, \ldots, a_n\}$, $B_n = B_n[D, K]$. Take any $a_{i_1}, a_{i_2} \in K$. Let γ_1, γ_2 be two different simple paths in $D - (K - a_{i_1} - a_{i_2})$ connecting a_{i_1} with a_{i_2}, let $H(\gamma_1), H(\gamma_2)$ be the half-twists corresponding to γ_1, γ_2, and let $\widetilde{H}(\gamma_1)$, $\widetilde{H}(\gamma_2)$ be the images of $H(\gamma_1)$, $H(\gamma_2)$ in $\widetilde{\widetilde{B}}_n$.

Lemma 7.2. *Let γ_1, γ_2 be two simple paths in $D - \{K - a_{i_1} - a_{i_2}\}$ connecting a_{i_1} with a_{i_2}. Then $\widetilde{H}(\gamma_1)^2 = \widetilde{H}(\gamma_2)^2$.*

Proof. Choose a frame of B_n $(Y_1, \ldots, Y_{n-1})$ such that $Y_1 = H(\gamma_1)$. Let $b \in B_n$ be such that $\gamma_2 = (\gamma_1)b$, that is, $H(\gamma_1)_b = H(\gamma_2)$. Let $\widetilde{Y}_i$ be the image of Y_i in $\widetilde{\widetilde{B}}_n$.

Let σ_1 be the image of b in S_n. Since $(a_i)b = a_i$, $(a_j)b = a_j$, $\sigma_1 \in \mathrm{Stab}(i) \cap \mathrm{Stab}(j)$ in S_n. The subgroup of $\widetilde{\widetilde{B}}_n$ generated by $\widetilde{Y}_3, \ldots, \widetilde{Y}_{n-1}$ is mapped by $\widetilde{\widetilde{\psi}}_n : \widetilde{\widetilde{B}}_n \to S_n$ onto $\mathrm{Stab}(i) \cap \mathrm{Stab}(j)$. Choose $\tilde{b}_1$ in this subgroup with its image in S_n equal to σ_1. Clearly, $(\widetilde{Y}_1)_{\tilde{b}_1} = \widetilde{Y}_1$. Since the image of $\tilde{b}_1^{-1}\tilde{b}$ in S_n is equal to $\sigma_1^{-1}\sigma_1 = \mathrm{Id}$, we have $\tilde{b}_1^{-1}\tilde{b} \in \widetilde{\widetilde{P}}_n$. Since $\widetilde{\widetilde{P}}_n$ is commutative when considering $\widetilde{Y}_1^2$ as an element of $\widetilde{\widetilde{P}}_n$, $(\widetilde{Y}_1^2)_{\tilde{b}_1^{-1}\tilde{b}} = \widetilde{Y}_1^2$. Thus, we have

$$\widetilde{H}(\gamma_2)^2 = \widetilde{H}(\gamma_1)^2_{\tilde{b}} = (\widetilde{Y}_1^2)_{\tilde{b}_1 \tilde{b}_1^{-1}\tilde{b}} = (\widetilde{Y}_1^2)_{\tilde{b}_1^{-1}\tilde{b}} = \widetilde{Y}_1^2 = \widetilde{H}(\gamma_1)^2. \qquad \square$$

Definition (f_{ij}). $\forall i,j \in (1,\ldots,n)$, $i \neq j$, we define $f_{ij} \in \tilde{\tilde{P}}_n$ as follows: take any simple path γ in $D-(K-a_i-a_j)$ connecting a_i with a_j. Let $f_{ij} = \widetilde{H}(\gamma)^2$. Lemma 7.2 shows that this definition does not depend on the choice of γ. We choose for $i<j$:

$$f_{ij} = \begin{cases} (\widetilde{X}_i^2)_{X_{i+1}\cdot\cdots\cdot X_{j-1}}, & 2<j-i, \\ \widetilde{X}_i^2, & i+1=j. \end{cases}$$

It is clear that for σ_1 the image in S_n of $b \in \widetilde{B}_n$ we have

$$(f_{ij})_b = f_{(i)\sigma_1,(j)\sigma_1}.$$

It is clear from our choice of γ for $f_{ij} = \widetilde{H}(\gamma)^2$ that

$$\psi_n(\widetilde{H}(\gamma)) = (i,j).$$

It will be convenient to use the following notation for $g \in G$ and $b \in \tilde{\tilde{B}}_n$.

Notation ($[g,b]$). For $g \in G$, $b \in \tilde{\tilde{B}}_n$ and the action of $\tilde{\tilde{B}}_n$ on G, we denote $[g,b] = g \cdot g_{b^{-1}}^{-1}$.

One can check that:

$$g_b = g \Leftrightarrow [g,b] = 1, \qquad [g,b]_z = g_z (g_z)_{b_z^{-1}}^{-1},$$

$$[g^{-1},b] = [g,b]^{-1}, \qquad [g,b^{-1}] = [g,b]_b^{-1},$$

$$[g_1g_2,b] = [g_2,b]_{g_1^{-1}} \cdot [g_1,b], \qquad [g,b_1b_2] = [g,b_1]\cdot[g,b_2]_{b_1^{-1}}.$$

Notation ($Q_{b,l,m}$). $\forall b \in \tilde{\tilde{B}}_n$, $\forall l,m \in (1,\ldots,n)$, $l \neq m$, we denote $Q_{b,l,m} = [S_b, f_{lm}^{-1}]$.

Lemma 7.3.

(i) *Let $b \in \tilde{\tilde{B}}_n$ be such that $(\{1,2\})\psi_n(b) \cap \{l,m\} = \emptyset$. Then $Q_{b,l,m} = 1$.*

(ii) *Let $Q = Q_{\mathrm{Id},1,3} = [S, f_{13}^{-1}]$. Then $Q_{\widetilde{X}_2^{-1}} = Q$.*

Proof.

(i) Let $\{l_1,m_1\} = (\{l,m\})\psi_n(b)^{-1}$. So $(f_{lm})_{b^{-1}} = f_{l_1m_1}$. We have

$$\{1,2\} \cap \{l_1,m_1\} = \emptyset,$$

that is, $3 \leqslant l_1$, $3 \leqslant m_1$. By our choice, f_{lm} is a product of X_j for $j \geqslant 3$. Thus, using property (3) of S we get $S_{f_{l_1,m_1}} = S$. In other words, $[S, f_{l_1,m_1}^{-1}] = 1$. We get

$$(Q_{b,l,m})_{b^{-1}} = [S_b, f_{lm}^{-1}]_{b^{-1}} = [S, (f_{lm}^{-1})_{b^{-1}}] = [S, f_{l_1,m_1}^{-1}] = 1,$$

and so $Q_{b,l,m} = 1$.

(ii) From $S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = S^{-1}S_{\widetilde{X}_2^{-1}}$ (assumption (1a) of Proposition 7.1) it follows that $S_{\widetilde{X}_2^{-1}} = SS_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}}$. Applying $\widetilde{X}_3^{-1}$, we get

$$S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = SS_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}} \tag{7.1}$$

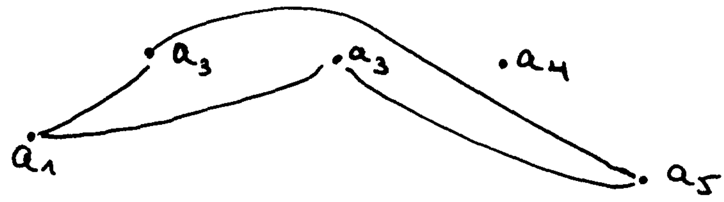


Fig. 11.

which, after applying $\widetilde{X}_2^{-1}$, gives

$$S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}} = S_{\widetilde{X}_2^{-1}} S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}}.$$

Since $S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}} = S_{\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}$, we obtain

$$S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = S_{\widetilde{X}_2^{-1}} S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}}, \qquad \text{or}$$

$$S_{\widetilde{X}_2^{-1}} = S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} S^{-1}_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}}. \tag{7.2}$$

Let $b_1 = \widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2^{-1}$. Observing that $(f_{13})_{\widetilde{X}_2^{-1}} = f_{12}$, we get from (7.2): $Q_{\widetilde{X}_2^{-1}} = [S_{\widetilde{X}_2^{-1}}, (f_{13}^{-1})_{\widetilde{X}_2^{-1}}] = [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} S_{b_1}^{-1}, f_{12}^{-1}]$. Thus

$$Q_{\widetilde{X}_2^{-1}} = [S_{b_1}^{-1}, f_{12}^{-1}]_{S^{-1}_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}} \cdot [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{12}^{-1}]. \tag{7.3}$$

Since $\psi_n(b_1) = (2\ 3)(1\ 2)(3\ 4)(2\ 3)$ (products of transpositions), $(\{1,2\})\psi_n(b_n) = \{3,4\}$. Since $\{3,4\} \cap \{1,2\} = \emptyset$, we get from (i) that $Q_{b_1,1,2} = 1$. Thus, $[S_{b_1}^{-1}, f_{12}^{-1}] = [S_{b_1}, f_{12}^{-1}]^{-1}_{S_{b_1}^{-1}} = (Q^{-1}_{b_1,1,2})_{S_{b_1}}$. (7.3) now gives

$$Q_{\widetilde{X}_2^{-1}} = [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{12}^{-1}]. \tag{7.4}$$

Consider a quadrangle formed by $\{a_1, a_2, a_3, a_5\}$, as in Fig. 11. By Lemma 1.2, we can write in $\widetilde{\widetilde{P}}_n$: $f_{35}f_{12} = f_{25}f_{13}$, or $f_{12} = f_{35}^{-1}f_{25}f_{13}$, $f_{12}^{-1} = f_{13}^{-1}f_{25}^{-1}f_{35}$.

From (7.4) we get

$$Q_{\widetilde{X}_2^{-1}} = [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{13}^{-1}f_{25}^{-1}f_{35}] = [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{13}^{-1}] \cdot [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{25}^{-1}f_{35}]_{f_{13}}. \tag{7.5}$$

Consider

$$\begin{aligned}
[S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{25}^{-1}f_{35}] &= [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{25}^{-1}] \cdot [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{35}]_{f_{25}} \\
&= Q_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1},2,5} \cdot [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{35}^{-1}]^{-1}_{f_{35}^{-1}f_{25}} \\
&= Q_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1},2,5} \cdot (Q^{-1}_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1},3,5})_{f_{35}^{-1}f_{25}}.
\end{aligned}$$

Since $\psi(\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}) = (2\ 3)(3\ 4)$, the images of $\{1,2\}$ under it are $\{1,4\}$. But $\{1,4\} \cap \{2,5\} = \emptyset$ and $\{1,4\} \cap \{3,5\} = \emptyset$. Thus, we get by (i) that

$$Q_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1},2,5} = Q_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1},3,5} = 1,$$

and so $[S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{25}^{-1}f_{35}] = 1$. (7.5) now implies $Q_{\widetilde{X}_2^{-1}} = [S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{13}^{-1}]$. By (7.1)

$$S_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = S \cdot S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}}$$

which gives

$$\begin{aligned} Q_{\widetilde{X}_2^{-1}} &= [S \cdot S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}}, f_{13}^{-1}] = [S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}}, f_{13}^{-1}]_{S^{-1}} \cdot [S, f_{13}^{-1}] \\ &= (Q_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-2}\widetilde{X}_3^{-1},1,3})_{S^{-1}} \cdot Q. \end{aligned}$$

The value of $\psi(\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1})$ $(= (2\ 3)(1\ 2)(3\ 4))$ on $\{1,2\}$ is $\{2,4\}$. Since $\{2,4\} \cap \{1,3\} = \emptyset$ we get from part (i) of the lemma that $Q_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}\widetilde{X}_3^{-1},1,3} = 1$, therefore,

$$Q_{\widetilde{X}_2^{-1}} = Q. \qquad \square$$

Lemma 7.4. $\tau = Q^{-1}$.

Proof. By the assumption on τ, $S_{\widetilde{X}_1^{-1}} = S^{-1}\tau$. By definition of T, $T_{\widetilde{X}_1^{-2}} = S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-2}}$. We apply assumption (1a) twice to get, using $S_{\widetilde{X}_1^{-1}} = S^{-1}\tau$, that

$$(S^{-1}S_{\widetilde{X}_2^{-1}})_{\widetilde{X}_1^{-1}} = S^{-1}_{\widetilde{X}_1^{-1}} S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = \tau^{-1}S \cdot S^{-1}S_{\widetilde{X}_2^{-1}} = \tau^{-1}S_{\widetilde{X}_2^{-1}} = \tau^{-1}T.$$

Thus

$$T_{\widetilde{X}_1^{-2}} = \tau^{-1}T, \quad \text{or} \quad \tau^{-1} = T_{\widetilde{X}_1^{-2}}T^{-1}. \tag{7.6}$$

Applying $\widetilde{X}_1^2$ on (7.6) and using $\tau_{\widetilde{X}_1^2} = \tau$ (assumption (2a)), we get

$$\begin{aligned} \tau^{-1} &= T \cdot T^{-1}_{\widetilde{X}_1^2} = [T, \widetilde{X}_1^{-2}] = [S_{\widetilde{X}_2^{-1}}, f_{12}^{-1}] = [S, (f_{12}^{-1})_{\widetilde{X}_2}]_{\widetilde{X}_2^{-1}} = [S, f_{13}^{-1}]_{\widetilde{X}_2^{-1}} \\ &= Q_{\widetilde{X}_2^{-1}} \overset{\text{Lemma 7.3}}{=} Q, \end{aligned}$$

that is,

$$\tau^{-1} = Q, \quad \text{or} \quad \tau = Q^{-1}. \qquad \square \tag{7.7}$$

Lemma 7.5. $\forall j \geqslant 3$, $\tau_{\widetilde{X}_j} = \tau$.

Proof. From $\tau = SS_{\widetilde{X}_1^{-1}}$ and $S_{\widetilde{X}_j} = S$ $\forall j \geqslant 3$ it follows that $\tau_{\widetilde{X}_j} = \tau$ $\forall j \geqslant 3$. $\square$

Lemma 7.6. $\tau_{\widetilde{X}_1} = \tau$.

Proof. Let us use now $\tau^{-1}_{\widetilde{X}_1} = \tau_T$ ((2b) of Proposition 7.1). By Lemmas 7.3 and 7.4 $\tau_{\widetilde{X}_2} = \tau$. Thus,

$$\tau_T = \tau_{S_{\widetilde{X}_2^{-1}}} = S^{-1}_{\widetilde{X}_2^{-1}} \tau S_{\widetilde{X}_2^{-1}} = (S^{-1}\tau S)_{\widetilde{X}_2^{-1}} = (\tau_S)_{\widetilde{X}_2^{-1}}.$$

So $\tau_{\widetilde{X}_1^{-1}} = \tau_{\widetilde{X}_1} = \tau_T^{-1} = (\tau_S^{-1})_{\widetilde{X}_2^{-1}}$, or

$$\tau_S = \tau^{-1}_{\widetilde{X}_1^{-1}\widetilde{X}_2}. \tag{7.8}$$

Since $\tau_{\widetilde{X}_3} = \tau$ and $S_{\widetilde{X}_3} = S$, we get $(\tau_S)_{\widetilde{X}_3} = \tau_S$ and

$$\tau_{\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_3} = (\tau_S^{-1})_{\widetilde{X}_3} = \tau_S^{-1} = \tau_{\widetilde{X}_1^{-1}\widetilde{X}_2}. \tag{7.9}$$

Applying $\widetilde{X}_2^{-1}$ on (7.9), we get $\tau_{\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_3\widetilde{X}_2^{-1}} = \tau_{\widetilde{X}_1^{-1}}$. Since $\tau_{X_3} = \tau$ and $\langle X_2, X_3\rangle = 1$, $\tau_{\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_3\widetilde{X}_2^{-1}} = \tau_{\widetilde{X}_1^{-1}\widetilde{X}_3^{-1}\widetilde{X}_2\widetilde{X}_3} = \tau_{\widetilde{X}_3^{-1}\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_3} = \tau_{\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_3}$. Thus

$$\tau_{\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_3} = \tau_{\widetilde{X}_1^{-1}}. \tag{7.10}$$

Combining formulas (7.9)–(7.10) we get $\tau_{\widetilde{X}_1^{-1}} = \tau_{\widetilde{X}_1^{-1}\widetilde{X}_2}$. Applying it to $\widetilde{X}_1$ we get $\tau = \tau_{\widetilde{X}_1^{-1}\widetilde{X}_2\widetilde{X}_1} = \tau_{\widetilde{X}_2\widetilde{X}_1\widetilde{X}_2^{-1}} = \tau_{\widetilde{X}_1\widetilde{X}_2^{-1}}$. Thus $\tau = \tau_{\widetilde{X}_1\widetilde{X}_2^{-1}}$, or $\tau_{\widetilde{X}_1} = \tau_{\widetilde{X}_2} = \tau$. □

Lemma 7.7. $\tau_{\widetilde{X}_j} = \tau\ \forall j = 1, 2, \ldots, n-1$.

Proof. By Lemmas 7.3–7.6. □

Lemma 7.8. $\tau_S = \tau^{-1}$.

Proof. From $\tau_{\widetilde{X}_1} = \tau_{\widetilde{X}_2} = \tau$ and (7.8). □

Lemma 7.9. $\tau_S = \tau$.

Proof. Consider assumption (1b) of Proposition 7.1

$$S_{\widetilde{X}_1\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = S^{-1}_{\widetilde{X}_1} S_{\widetilde{X}_1\widetilde{X}_2^{-1}}.$$

Using $S_{\widetilde{X}_1^{-1}} = S^{-1}\tau$ and $\tau_{\widetilde{X}_1} = \tau$, we get $S = S^{-1}_{\widetilde{X}_1}\tau$, or $S^{-1}_{\widetilde{X}_1} = S\tau^{-1}$, $S_{\widetilde{X}_1} = \tau S^{-1}$. Assumption (1b) now gives (using $\tau_{\widetilde{X}_i} = \tau\ \forall i = 1, \ldots, n-1$)

$$\tau S^{-1}_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = S\tau^{-1} \cdot \tau S^{-1}_{\widetilde{X}_2^{-1}} = SS^{-1}_{\widetilde{X}_2^{-1}} = ST^{-1}.$$

On the other hand, by (1a) and (2), $S_{\widetilde{X}_2^{-1}\widetilde{X}_1^{-1}} = S^{-1}T$. Thus

$$\tau S^{-1}_{X_2^{-1}} X_1^{-1} = \tau T^{-1} S.$$

We compare the last two expressions to get $\tau T^{-1}S = ST^{-1}$, or

$$\tau = ST^{-1}S^{-1}T, \quad \text{or} \quad T^{-1}_{S^{-1}} = \tau T^{-1}. \tag{7.11}$$

By Lemmas 7.4 and 7.7

$$Q = Q_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = \left[S_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, (f_{13}^{-1})_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}\right].$$

Thus

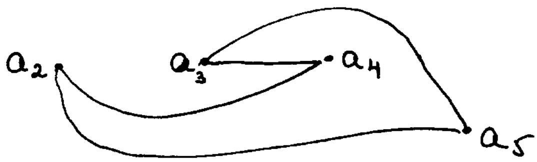


Fig. 12.

$$Q = \left[S_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{24}^{-1}\right] \tag{7.12}$$

(we use $(\{1,3\})\psi(\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}) = (\{1,3\})(1\ 2)(2\ 3)(3\ 4) = \{4,2\})$.

Considering a quadrangle formed by a_2, a_3, a_4, a_5 (see Fig. 12) we can write in $\widetilde{\widetilde{P}}_n$ (Lemma 2.2) $f_{35}f_{24} = f_{25}f_{34}$, or $f_{24} = f_{35}^{-1}f_{25}f_{34}$, $f_{24}^{-1} = f_{34}^{-1}f_{25}^{-1}f_{35}$. From (7.12) we get, denoting by $b = \widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}$,

$$\begin{aligned} Q &= \left[S_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{24}^{-1}\right] = \left[S_b, f_{34}^{-1}f_{25}^{-1}f_{35}\right] \\ &= \left[S_b, f_{34}^{-1}\right]\left[S_b, f_{25}^{-1}f_{35}\right]_{f_{34}} = Q_{b,3,4} \cdot \left[S_b, f_{25}^{-1}\right]_{f_{34}} \cdot [S_b, f_{35}]_{f_{25}f_{34}} \\ &= Q_{b,3,4} \cdot (Q_{b,2,5})_{f_{34}} \cdot \left[S_b, f_{35}^{-1}\right]^{-1}_{f_{35}^{-1}f_{25}f_{34}} = Q_{b,3,4} \cdot (Q_{b,2,5})_{f_{34}} \cdot (Q_{b,3,5})_X^{-1}. \end{aligned} \tag{7.13}$$

Now, $(\{1,2\})\psi(b) = \{1,2\}(1\ 2)(2\ 3)(3\ 4) = \{4,1\}$. Since $\{4,1\} \cap \{2,5\} = \emptyset$ and $\{4,1\} \cap \{3,5\} = \emptyset$, we get by Lemma 7.3 that $Q_{b,2,5} = Q_{b,3,5} = 1$, and by (7.13)

$$Q = Q_{b,3,4}.$$

We can write

$$S_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = \left(S^{-1}\tau\right)_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}} = S^{-1}_{\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}\tau = T^{-1}_{\widetilde{X}_3^{-1}}\tau$$

(using $\tau_{\widetilde{X}_i} = \tau\ \forall i$). So

$$\begin{aligned} Q = Q_{b,3,4} &= \left[S_{\widetilde{X}_1^{-1}\widetilde{X}_2^{-1}\widetilde{X}_3^{-1}}, f_{34}^{-1}\right] \overset{f_{34}=\widetilde{X}_3^2}{=} \left[T^{-1}_{\widetilde{X}_3^{-1}}\tau, \widetilde{X}_3^{-2}\right] \\ &= \left[\tau, \widetilde{X}_3^{-2}\right]_{T_{\widetilde{X}_3^{-1}}} \cdot \left[T^{-1}_{\widetilde{X}_3^{-1}}, \widetilde{X}_3^{-2}\right] = \left[T^{-1}_{\widetilde{X}_3^{-1}}, \widetilde{X}_3^{-2}\right] = \left[T^{-1}, \widetilde{X}_3^{-2}\right]_{\widetilde{X}_3^{-1}}. \end{aligned}$$

Since $Q_{\widetilde{X}_3} = Q$, we get

$$Q = \left[T^{-1}, \widetilde{X}_3^{-2}\right]. \tag{7.14}$$

This implies

$$\begin{aligned} Q_{S^{-1}} &= \left(T^{-1}T_{\widetilde{X}_3^2}\right)_{S^{-1}} \overset{\text{assumption (3)}}{=} T^{-1}_{S^{-1}} \cdot (T_{S^{-1}})_{\widetilde{X}_3^2} = \left[T^{-1}_{S^{-1}}, \widetilde{X}_3^{-2}\right] \\ &\overset{(7.11)}{=} \left[\tau T^{-1}, \widetilde{X}_3^{-2}\right] = \left[T^{-1}, \widetilde{X}_3^{-2}\right]_{\tau^{-1}}\left[\tau, \widetilde{X}_3^{-2}\right] \overset{\text{Lemma 7.5}}{=} \left[T^{-1}, \widetilde{X}_3^{-2}\right]_{\tau^{-1}} \\ &\overset{(7.14)}{=} Q_{\tau^{-1}}. \end{aligned}$$

Using $Q = \tau^{-1}$ we get $\tau^{-1}_{S^{-1}} = \tau^{-1}_{\tau^{-1}}$. Thus, $\tau_{S^{-1}} = \tau$ and $\tau_S = \tau$. □

We can now finish the proof of Proposition 7.1.

By Lemma 2.4, we only have to prove that $\tau^2 = 1$, $\tau_b = \tau\ \forall b \in \tilde{\tilde{B}}_n$ and $\tau \in \mathrm{Center}(G)$.

By Lemma 7.9, $\tau_S = \tau$, and by Lemma 7.8, $\tau_S = \tau^{-1}$. Thus, $\tau = \tau^{-1}$ and $\tau^2 = 1$.

By Lemma 7.7, $\tau_{\tilde{X}_i} = \tau\ \forall i \in (1, \dots, n-1)$. Thus $\tau_b = \tau\ \forall b \in \tilde{\tilde{B}}_n$.

By Lemma 7.9, $\tau_S = \tau$, i.e., $[\tau, S] = 1$. Let $b \in \tilde{B}_n$, $[\tau, S_b] = [\tau_{b^{-1}}, S]_b = [\tau, S]_b = 1$. Thus τ commutes with $S_b\ \forall b \in \tilde{\tilde{B}}_n$. Since $\ker(\tilde{B}_n \to \tilde{\tilde{B}}_n)$ acts trivially on G, $\tilde{\tilde{B}}_n$ acts on G via $\tilde{B}_n$, and thus τ commutes with $S_b\ \forall b \in \tilde{B}_n$.

By assumption (0) of Proposition 7.1, G is generated by $\{S_b\}_{b \in \tilde{B}_n}$. Thus

$\tau \in \mathrm{Center}(G)$. □

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